Annihilation Attacks for Multilinear Maps

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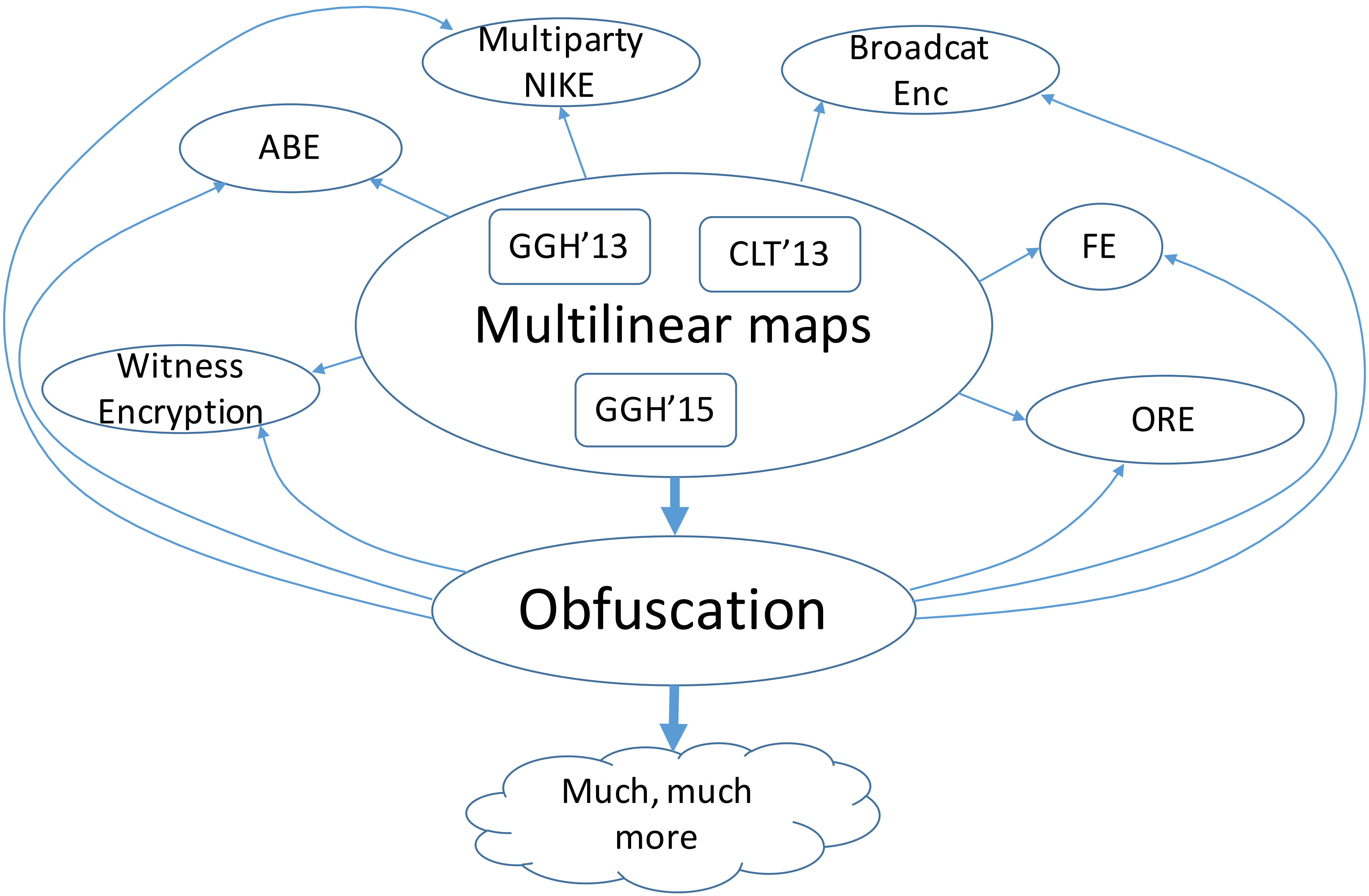
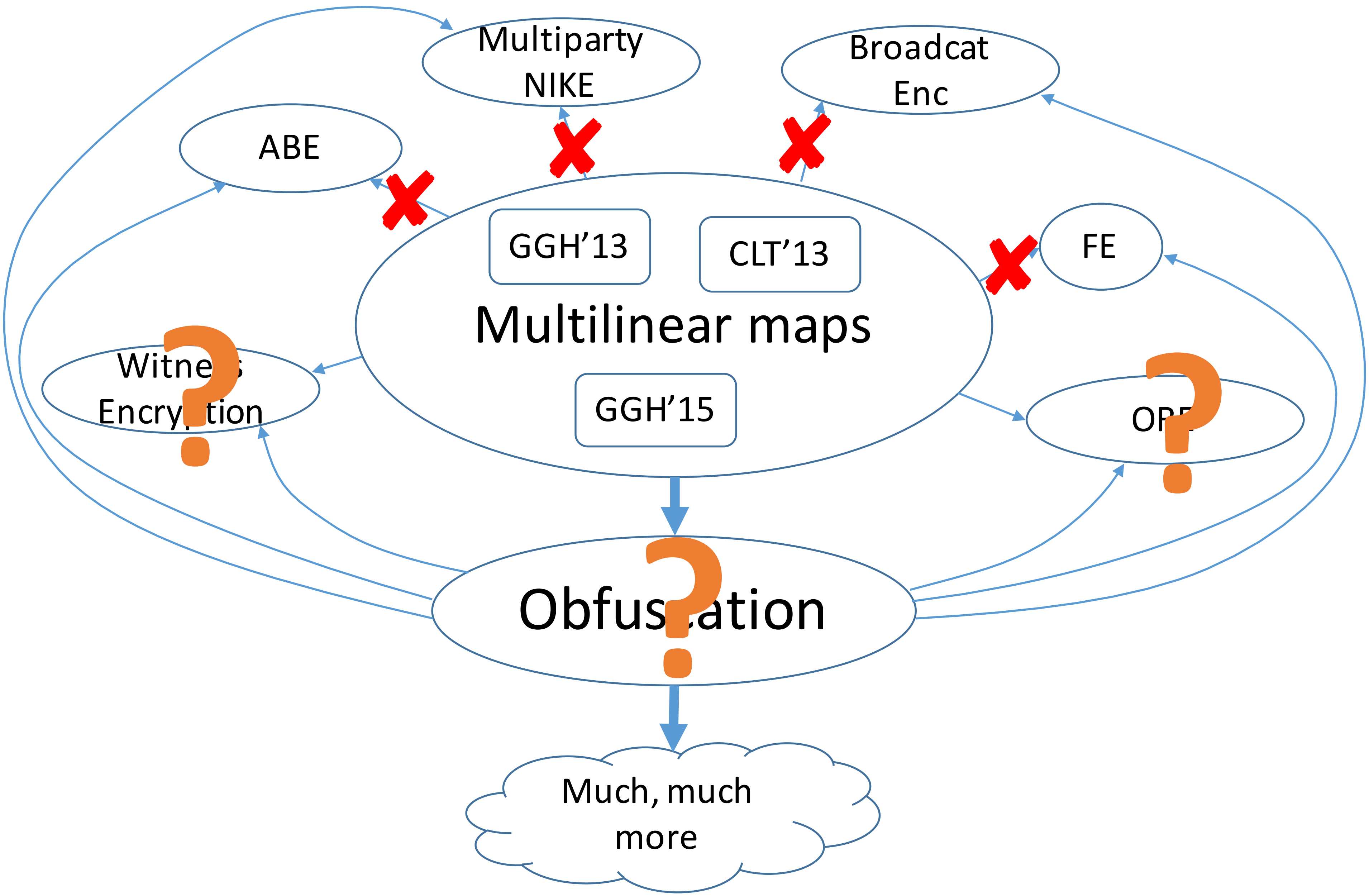
Multilinear Maps



Mmap Attacks (GGH'13a, CHLRS'15, BWZ'14, CGHLMMR'15, HJ'15, BGHLST'15, Hal'15, CLR'15, MF'15, CLLT'15, CFLMR'16)



This Work

Goal: Understand if/why Obfuscation/Witness Enc/ORE actually resists attack

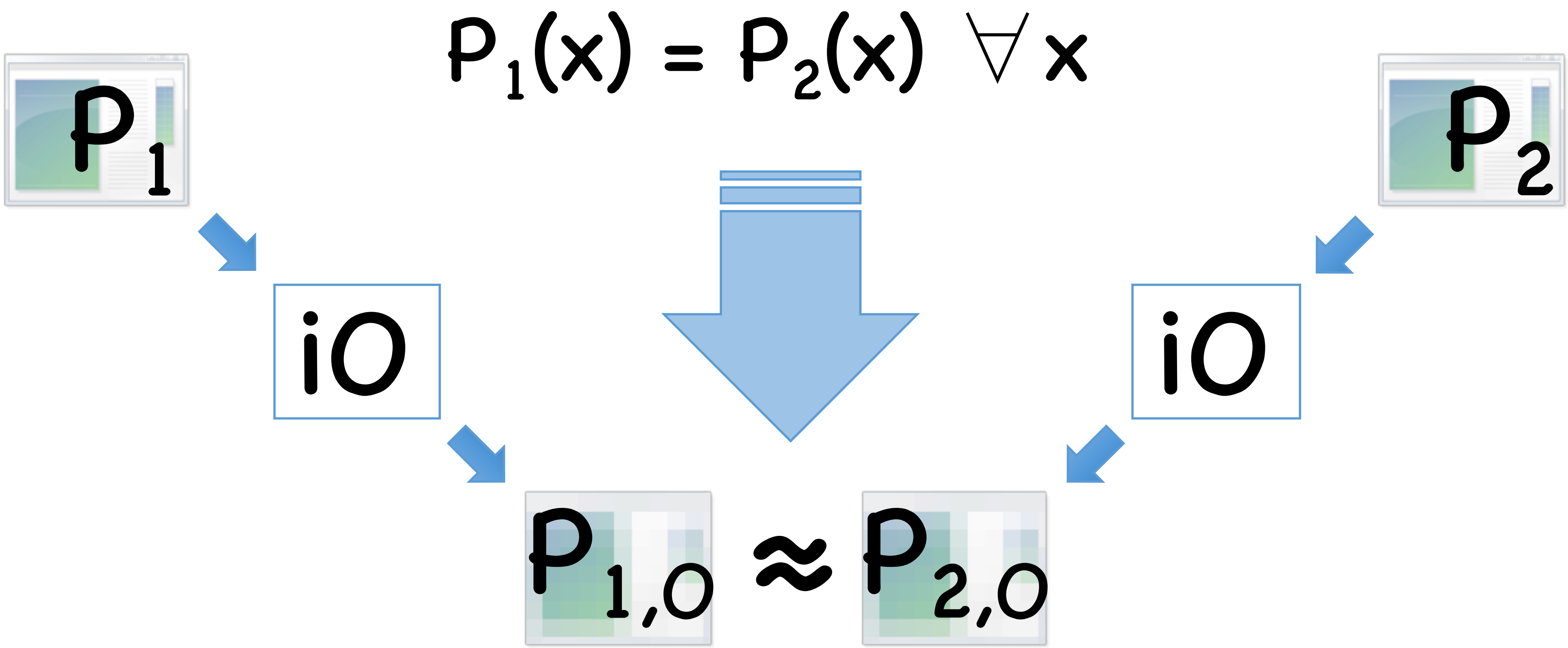
Background…

Obfuscation

Intuition: scramble a program

- Maintain functionality, hide implementation

"Industry accepted" security notion: iO



High-level description GGH13

Level i encoding of x: $\dfrac{\overbrace{x + g\,r}^{\text{“short”}}}{z^i}$

- Add within levels: $\dfrac{x_1+gr_1}{z^i} + \dfrac{x_2+gr_2}{z^i} = \dfrac{(x_1+x_2)+g(r_1+r_2)}{z^i}$

- Multiply: $\dfrac{x_1+gr_1}{z^i} \cdot \dfrac{x_2+gr_2}{z^j} = \dfrac{(x_1x_2)+g(r_1x_2+r_2x_1+gr_1r_2)}{z^{i+j}}$

IsZero(level k encoding e): test if $p_{zt}e$ is “not too big”

- $p_{zt} = \dfrac{h\ z^k}{g}$ “not too big”

$$p_{zt}\frac{gr}{z^k} = hr \quad \text{“not too big”}$$

“big”

$$p_{zt}\frac{x+gr}{z^k} = \frac{hx}{g} + hr$$

High-level description GGH13

Level i encoding of x: $\dfrac{x + g\,r}{z^i}$

IsZero(level k encoding e): test if $p_{zt}e$ is "not too big"

Intuition:

- Can eval arbitrary degree-k polys on level-1 encodings, then zero test
- For any degree higher than k, zero test gives junk

For obfuscation, use "asymmetric" variant

- Enforces additional constraints on allowable polys

Background on Mmap Attacks

For current mmaps, when IsZero=True, also get algebraic element **hr** that can be manipulated

- **r** may contain info about plaintexts

Idea behind all known (classical) attacks:

- Generate several "related" zero encodings
- Manipulate top-level encodings to learn non-trivial information

All attacks respect level restrictions

Background on Mmap Attacks

Prior attacks:

- Generally require some “low-level” zero encodings

  $\Rightarrow$ multiply together to get “related” zero encodings

- Extends to cases where no explicit low level zero encodings are given, but “effective” encodings of zero are given

- $\exists$ quantum attacks that don’t need low-level zeros [BS’15]

Background on Mmap Attacks

Most applications need low level encodings

- Used for “rerandomization”
- Required by most applications
- Hence, these applications broken

Also required to use most assumptions

- Inc. those used to build iO (e.g. [GLSW’14])
- Proofs often broken as well, even if application isn’t

Background on Mmap Attacks

Attacking obfuscation/witness encryption/ORE?

- No explicit low level zero encodings needed for schemes
- Top level zero encodings may still be generated during use

What next? Either:

1. Completely break application?
2. Argue that application is secure?
    - All known reductions to "simple" assumptions require low-level zero encodings
    - Need alternate way to argue security

Restricted Black Box Fields

F = Field, **P** = class of polynomials on **n** variables

$p \in \mathbf{P}$ →

← IsZero($p(a_1, a_2, \ldots, a_n)$)

$a_1, a_2, \ldots, a_n \in \mathbf{F}$

Generic Groups*: P = { Linear functions }

Black Box Fields*: P = { Polys with small algebraic circuits}

Symmetrix multilinear maps*: P = { Degree **k** polynomials}

Asymmetric multilinear maps*: P = { More complicated restrictions}

* Often need greater functionality requirements for protocols. This model suffices for our discussion

Obfuscation in Restricted BBFs

(model used by [BR’14,BGKPS’14,AGIS’14,Z’15,AB’15,BMSZ’16])

Obfuscate(C):

BBF with restricted polynomial class **P**

$p_x \in \mathbf{P}$

$a_1, a_2, \ldots, a_n \leftarrow D_C$

Eval(x):

IsZero($p(a_1, a_2, \ldots, a_n)$)

- If **IsZero** gives “True”, output 1
- If **IsZero** gives “False”, output 0

Unfortunately, restricted BBF does not capture mmap attacks

Refined Abstract Model for Mmap attacks

$p \in \mathbf{P}$

Efficient polys

$q \in \mathbf{Q}$

$a_1, a_2, \ldots, a_n \in \mathbf{F}$

$r_1, r_2, \ldots, r_n \leftarrow\$ \mathbf{F}$

indeterminant

Write $p(a_1+gr_1, \ldots, a_n+gr_n)$
$= c + dg + \ldots$

If $c \neq 0$, output "False"

If $c = 0$, output "True", $\mathbf{d}$

Unrestricted BBF

$d_1\ d_2\ d_3\ \ldots$

Refined Abstract Model for Mmap attacks

- Seems to capture intuition behind attacks

Proof in refined model → Heuristic evidence of security against current attacks

But keep in mind that:

Attack in refined model → Attack on actual protocol

Not trivially

Prior work: obfuscation for evasive functions [MBSZ'16]

What if function being obfuscated is evasive?

- When running obfuscator on any point the adversary can come up with, **IsZero** always gives "False"
- [BMSZ'16]: The only way to get IsZero to be "True" is through honest executions
- For evasive functions, all attacks apparently blocked
- In particular, witness enc secure against known attacks

What about non-evasive settings?

Attacking Obfuscation [MSZ'16]

Thm: The branching program obfuscators in [BGKPS'14, PST'14, AGIS'14, BMSZ'16] do not satisfy iO in the refined abstract model

Also: translate abstract model attack into concrete attack when instantiated using GGH'13 mmaps

- Small heuristic component

(Single input) Branching Programs

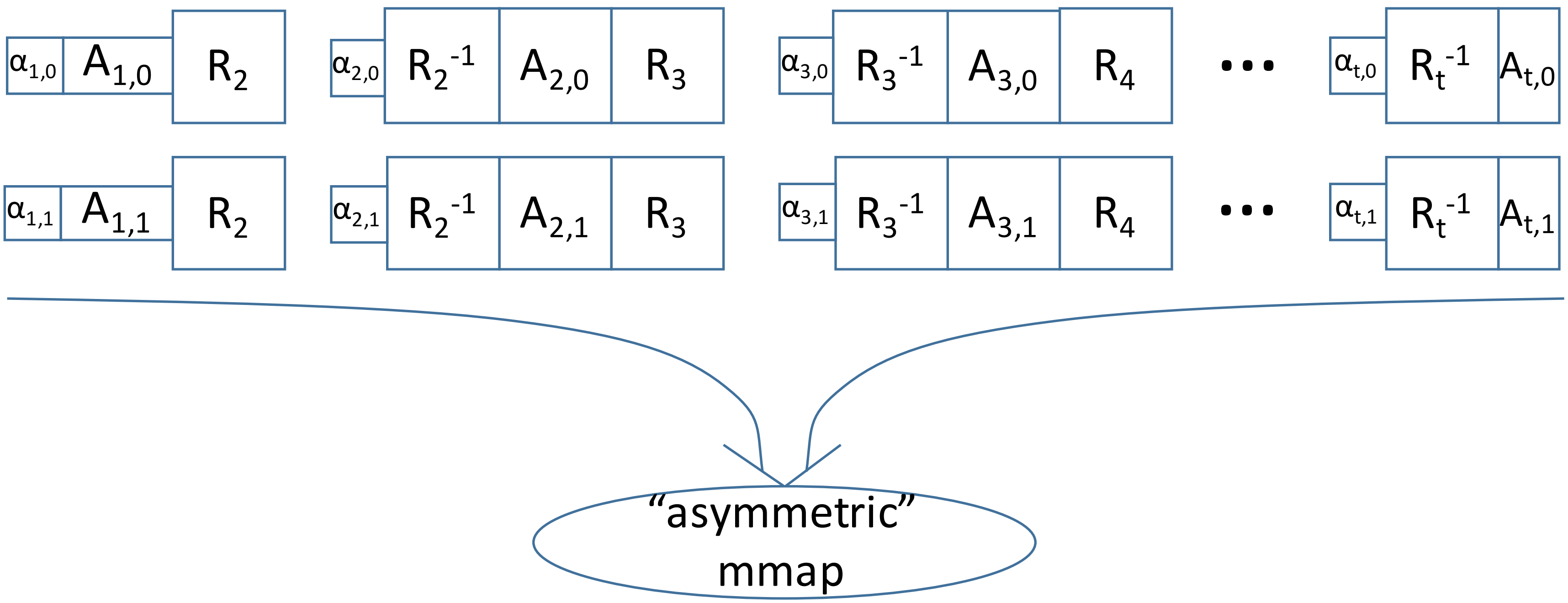
$A_{1,0}$	$A_{2,0}$	$A_{3,0}$	$A_{4,0}$	$A_{5,0}$	$A_{6,0}$	$A_{7,0}$	$A_{8,0}$
$A_{1,1}$	$A_{2,1}$	$A_{3,1}$	$A_{4,1}$	$A_{5,1}$	$A_{6,1}$	$A_{7,1}$	$A_{8,1}$
2	4	2	1	3	4	2	5

$x = 11001$:

$$p_x = \boxed{A_{1,1}}\,\boxed{A_{2,0}}\,\boxed{A_{3,1}}\,\boxed{A_{4,1}}\,\boxed{A_{5,0}}\,\boxed{A_{6,0}}\,\boxed{A_{7,1}}\,\boxed{A_{8,1}}$$

If $p_x = 0$, output 1, otherwise output 0

[BMSZ'16] Obfuscator



Thm ([BMSZ'16]): If level structure respected, only poly's that evaluate to 0 correspond to honest evaluations*

* Assuming mild technical condition on BP

[BMSZ'16] In Abstract Model

$B_{i,b} = \alpha_{i,b} \; R_i^{-1} \; A_{i,b} \; R_{i+1}$

Honest subset products

$p_x \in \mathbf{P}$

$S_{i,b} \leftarrow\$ \; \mathbf{F}$

Write $p_x(\ \{B_{i,b} + g\ S_{i,b}\}\ )$
$= c + dg + \ldots$

If $c \neq 0$, output "False"

If $c = 0$, d

Efficient polys

$q \in \mathbf{Q}$

Unrestricted BBF

$d_1 \;\; d_2 \;\; d_3 \;\; \ldots$

Attack: Annihilating Polynomials

- All terms are rational functions in underlying randomness
  $\Rightarrow$ each **d** is rational in underlying randomness
- Efficiency $\Rightarrow$ only poly-many free variables
- Exponentially many inputs $\Rightarrow$ exponentially many **d**
- Must be algebraic dependence among **d**

$$\exists \text{ poly } \mathbf{q}: \ \mathbf{q}(\mathbf{d}_1, \mathbf{d}_2, \ldots) = 0$$

- **q** will most likely depend on exact program obfuscated

Argument extends to any "purely algebraic" obfuscator

Attack: Annihilating Polynomials

q are called "annihilating polynomials"

Goal: find annihilating polynomials for various programs

Problem: in general, annihilating polys hard to compute

Thm ([Kay'09]): Unless PH collapses, there are dependent rational functions for which the annihilating polynomial requires super-polynomial sized circuits

Question: Can annihilating polys be found for particular obfuscators/programs?

Step 1: Variable Renaming

$$\boxed{B_{i,b}} + \boxed{g}\boxed{S_{i,b}} = \boxed{\alpha_{i,b}}\boxed{R_i^{-1}}\boxed{A_{i,b}}\boxed{R_{i+1}} + \boxed{g}\boxed{S_{i,b}}$$

$$= \boxed{\alpha_{i,b}}\boxed{R_i^{-1}}\left(\boxed{A_{i,b}} + \boxed{g}\boxed{T_{i,b}}\right)\boxed{R_{i+1}} \qquad \left(\boxed{T_{i,b}} = \boxed{\alpha_{i,b}^{-1}}\boxed{R_i}\boxed{S_{i,b}}\boxed{R_{i+1}^{-1}}\right)$$

For honest subset product polynomials, R_i's will cancel out

$$\Rightarrow p_x\left(\boxed{B_{i,b}} + \boxed{g}\boxed{S_{i,b}}\right) = p_x\left(\boxed{A_{i,b}} + \boxed{g}\boxed{T_{i,b}}\right)$$

Step 2: Look at g^1 Coefficient

Coefficient of g^1 in $p_x\left(\boxed{A_{i,b}} + \boxed{g}\boxed{T_{i,b}}\right)$:

$$d_x = \boxed{\beta_x}\boxed{T_{1,1}}\boxed{A_{2,0}}\boxed{A_{3,1}}\boxed{A_{4,1}}\boxed{A_{5,0}}\boxed{A_{6,0}}\boxed{A_{7,1}}\boxed{A_{8,1}} +$$

$$\boxed{\beta_x}\boxed{A_{1,1}}\boxed{T_{2,0}}\boxed{A_{3,1}}\boxed{A_{4,1}}\boxed{A_{5,0}}\boxed{A_{6,0}}\boxed{A_{7,1}}\boxed{A_{8,1}} +$$

$$\boxed{\beta_x}\boxed{A_{1,1}}\boxed{A_{2,0}}\boxed{T_{3,1}}\boxed{A_{4,1}}\boxed{A_{5,0}}\boxed{A_{6,0}}\boxed{A_{7,1}}\boxed{A_{8,1}} +$$

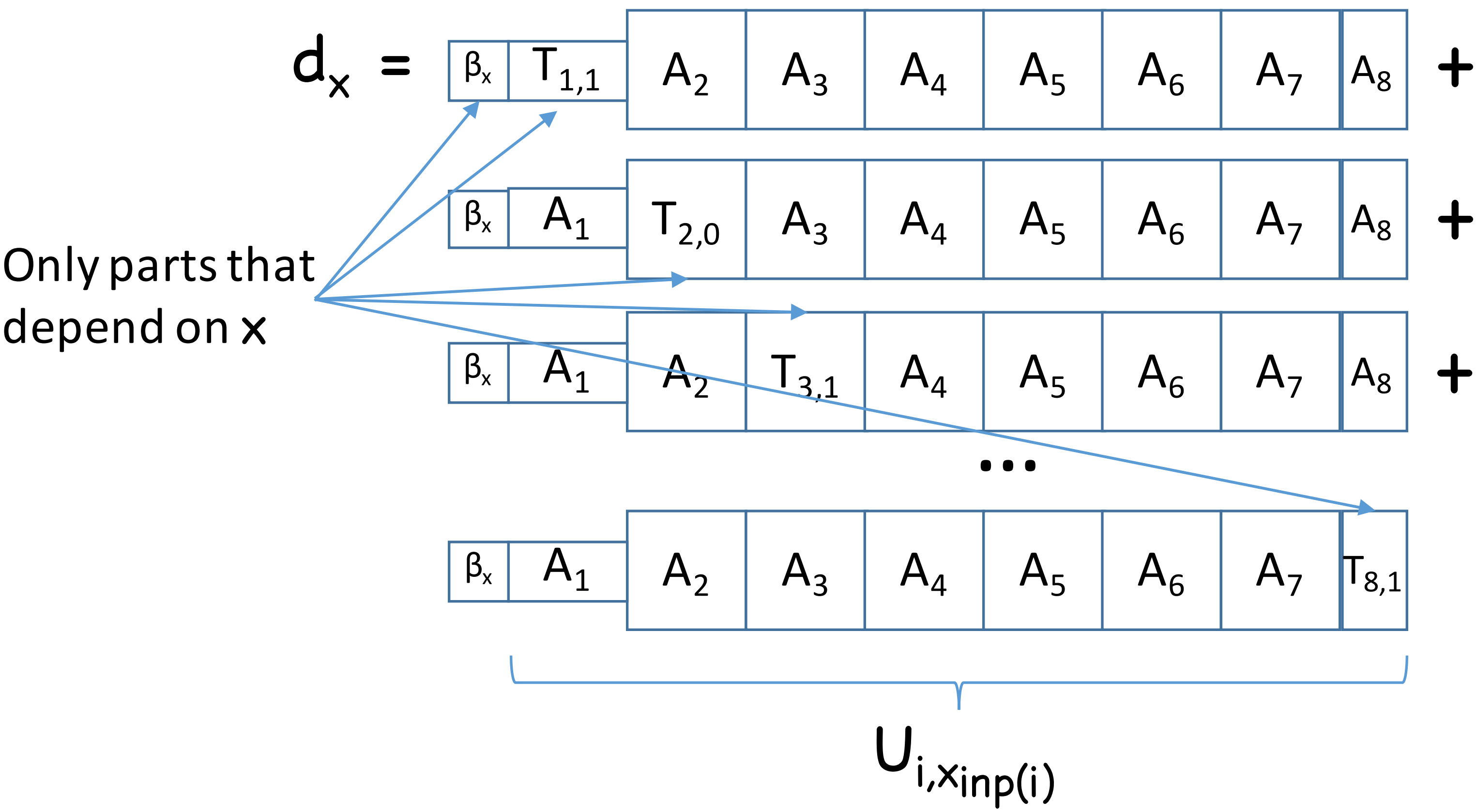
$$\cdots$$

$$\boxed{\beta_x}\boxed{A_{1,1}}\boxed{A_{2,0}}\boxed{A_{3,1}}\boxed{A_{4,1}}\boxed{A_{5,0}}\boxed{A_{6,0}}\boxed{A_{7,1}}\boxed{T_{8,1}}$$

$$\beta_x = \boxed{\alpha_{1,1}}\boxed{\alpha_{2,0}}\boxed{\alpha_{3,1}}\boxed{\alpha_{4,1}}\boxed{\alpha_{5,0}}\boxed{\alpha_{6,0}}\boxed{\alpha_{7,1}}\boxed{\alpha_{8,1}}$$

Step 2: Look at g^1 Coefficient

Suppose "trivial" branching program: $A_{i,0}=A_{i,1}=A_i$



Step 3: More Variable Renaming

Suppose "trivial" branching program: $A_{i,0}=A_{i,1}=A_i$

$$d_x = \beta_x \underbrace{(U_{1,x_2} + U_{2,x_4} + U_{3,x_2} + U_{4,x_1} + U_{5,x_3} + U_{6,x_4} + U_{7,x_2} + U_{8,x_5})}_{\gamma_x}$$

Collect **U** that read same input bit:

$$\boxed{\gamma_x = V_{1,x_1} + V_{2,x_2} + V_{3,x_3} + V_{4,x_4} + V_{5,x_5}}$$

Same treatment for β_x:

$$\boxed{\beta_x = W_{1,x_1} W_{2,x_2} W_{3,x_3} W_{4,x_4} W_{5,x_5}}$$

Step 4: Even More Variable Renaming

$$\gamma_x = V_{1,x_1} + V_{2,x_2} + V_{3,x_3} + V_{4,x_4} + V_{5,x_5}$$

Linear algebra!

Position $\mathbf{i}$

$$e(i) = 0\ldots010\ldots0 \qquad\qquad 0 = 0^n$$

$$x \leq y:\ x_i=1 \Rightarrow y_i=1$$

$$\gamma_x = \sum_{e(i) \leq x} \gamma_{e(i)} - (|x|-1)\,\gamma_0$$

Now algebraic dependence is local

(E.g. can consider x that are non-zero at only first k bits)

Step 4: Even More Variable Renaming

$$\gamma_x = \sum_{e(i) \leq x} \gamma_{e(i)} - |x|\, \gamma_0 \qquad \beta_x = \prod_{e(i) \leq x} \gamma_{e(i)} \,/\, \beta_0^{|x|}$$

$$d_x = \beta_x \gamma_x$$

Consider **x** that are non-zero only in first **k** bits

- $\mathbf{2^k}$ different $\mathbf{d_x}$
- ~~**2(k+1)**~~ **2k+1** degrees of freedom
- For **k≥3**, must be algebraic dependence

Step 5: Brute Force Search

Brute force search for annihilating poly

$$\begin{aligned}\mathbf{q} = {} & (d_{000}d_{111})^2 + (d_{001}d_{110})^2 + (d_{010}d_{101})^2 + (d_{100}d_{011})^2 \\ & - 2d_{000}d_{111}d_{001}d_{110} - 2d_{000}d_{111}d_{010}d_{101} - 2d_{000}d_{111}d_{100}d_{011} \\ & - 2d_{001}d_{110}d_{010}d_{101} - 2d_{001}d_{110}d_{100}d_{011} - 2d_{010}d_{101}d_{100}d_{011} \\ & + 4d_{000}d_{011}d_{101}d_{111} + 4d_{111}d_{001}d_{010}d_{100}\end{aligned}$$

Annihilation very particular to “trivial” program

- **q** will not annihilate on “most” programs

The Abstract Attack

- Branching programs:
  - "Trivial" program that always outputs **1**
  - Non-trivial program that always outputs **1**

$p_{000}, p_{001}, p_{010}, \ldots \longrightarrow$

$\longleftarrow$ True, True, True, …

$$p_x(\{B_{i,b} + g\,S_{i,b}\}) = d_x g + \ldots$$

$q \longrightarrow$

$\longleftarrow$ True/False

Unrestricted BBF

$d_{000}, d_{001}, d_{010}, \ldots$

Extending to GGH'13 Candidate

Unfortunately, cannot directly test if **q=0** in GGH'13

- If **q** annihilates, obtain element in ideal **<g>**
- **<g>** hidden, so cannot immediately test membership

Our attack:

- Evaluate **q** on many sets of inputs $\mathbf{S}_j$
- Set up non-trivial program so that **q=0** for each $\mathbf{S}_j$

  $\Rightarrow$ Obtain many $\mathbf{x}_j$ in **<g>**, regardless of program
- Heuristically assume $\mathbf{x}_j$ span **<g>**
- Evaluate **q** on "test" set $\mathbf{S^*}$
  - **q** annihilates on $\mathbf{S^*}$ iff trivial program
- Test if result is in **<g>** using the $\mathbf{x}_i$

Further Extensions

So far, only discussed single input BMSZ’16

For dual input:

- Using same ideas, can reduce search to finite-size
- Can brute force annihilating polynomial in constant time
- Hasn’t found it yet… but still gives poly-time attack

Other obfuscators:

- [BGKPS’14, PST’14, AGIS’14]: similar analysis
- Particular attack fails for [GGHRSW’13]

Also attack ORE [BLRSZZ’15] over GGH’13

Takeaways

Old attacks: intuition about mmap security wrong
- Old abstract mmap invalid in presence of low-level zeros

Our attacks: intuition for obfuscation security (no low-level zeros) also wrong
- Old abstract mmap invalid even without low-level zeros
- Need to revisit all constructions using mmaps
- Need new ways to argue security

Future Work
- Extend attacks to other mmaps/obfuscators
- Defenses

Obfuscation