COS433/Math 473: Cryptography

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Spring 2020

Reminders

HW3 Due March 5th

PR1 Due **March 12th**

- No late days

Previously on COS 433…

Pseudorandom Permutations
(also known as block ciphers)

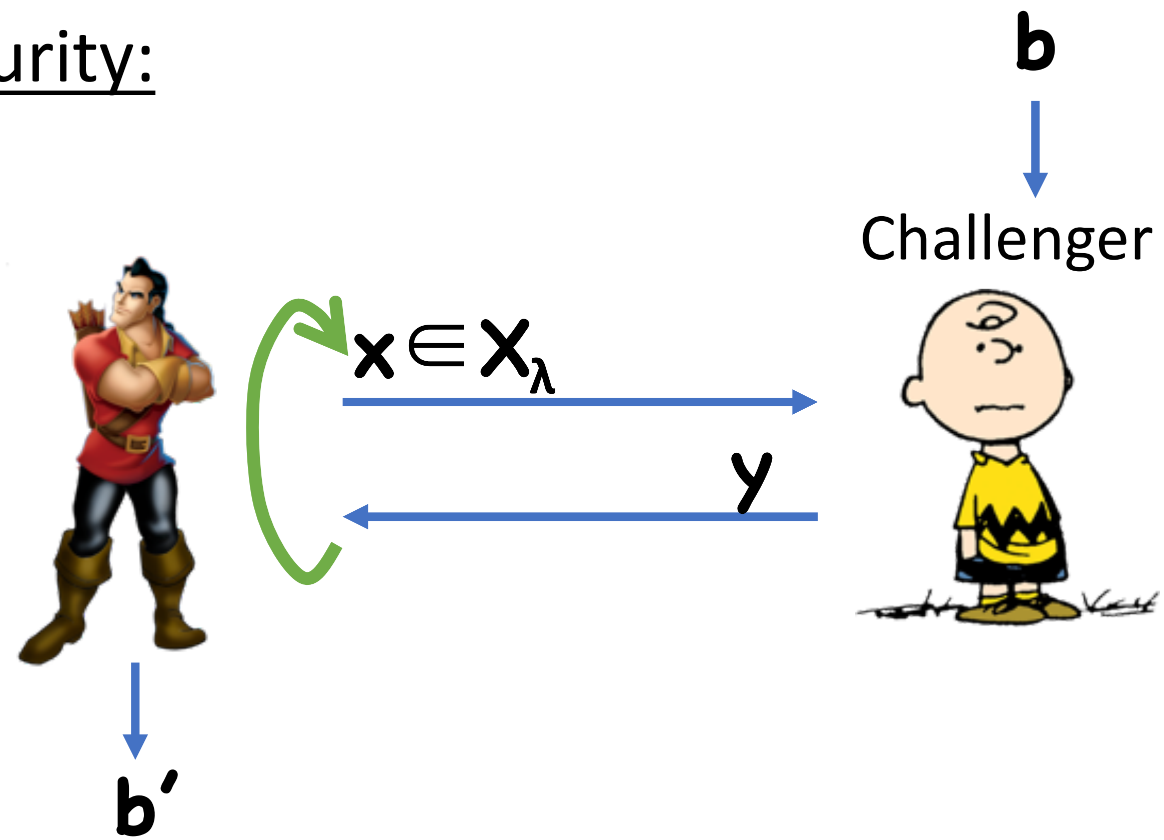
Functions that "look like" random **permutations**

Syntax:
- Key space $\mathbf{K}_\lambda$
- Domain=Range= $\mathbf{X}_\lambda$
- Function $\mathbf{F}:\mathbf{K}_\lambda \times \mathbf{X}_\lambda \rightarrow \mathbf{X}_\lambda$
- Function $\mathbf{F}^{-1}:\mathbf{K}_\lambda \times \mathbf{X}_\lambda \rightarrow \mathbf{X}_\lambda$

Correctness: $\forall \mathbf{k},\mathbf{x},\ \mathbf{F}^{-1}(\mathbf{k},\ \mathbf{F}(\mathbf{k},\ \mathbf{x})\ ) = \mathbf{x}$

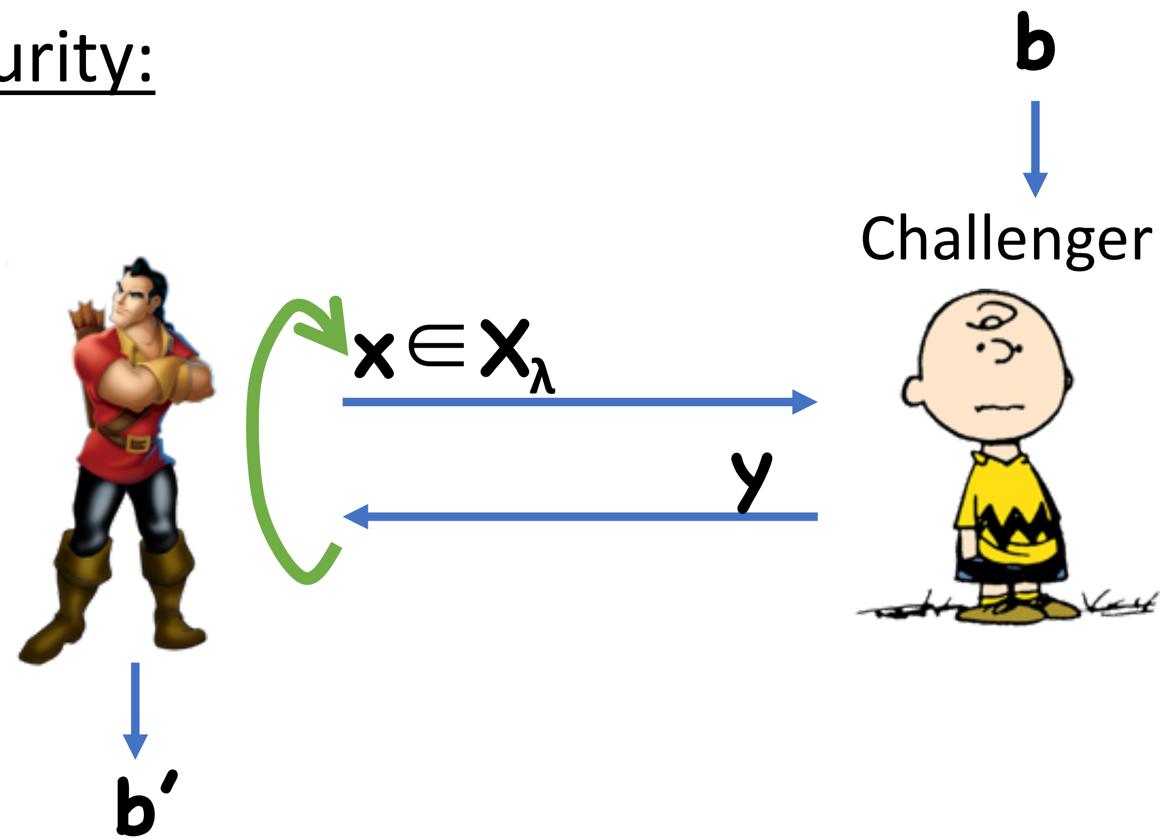
Pseudorandom Permutations

Security:



Pseudorandom Permutations

<u>Security:</u>



Pseudorandom Permutations

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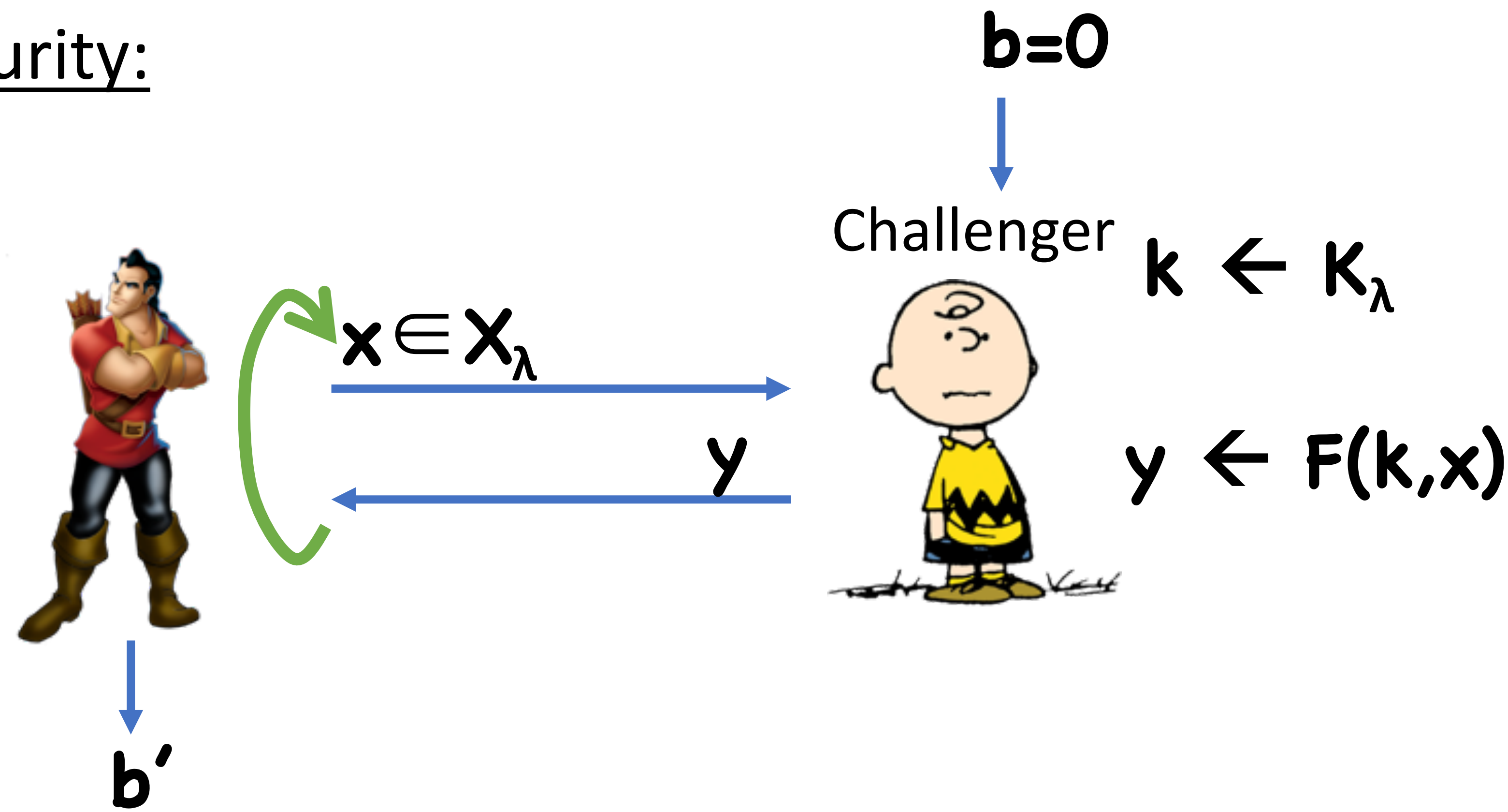
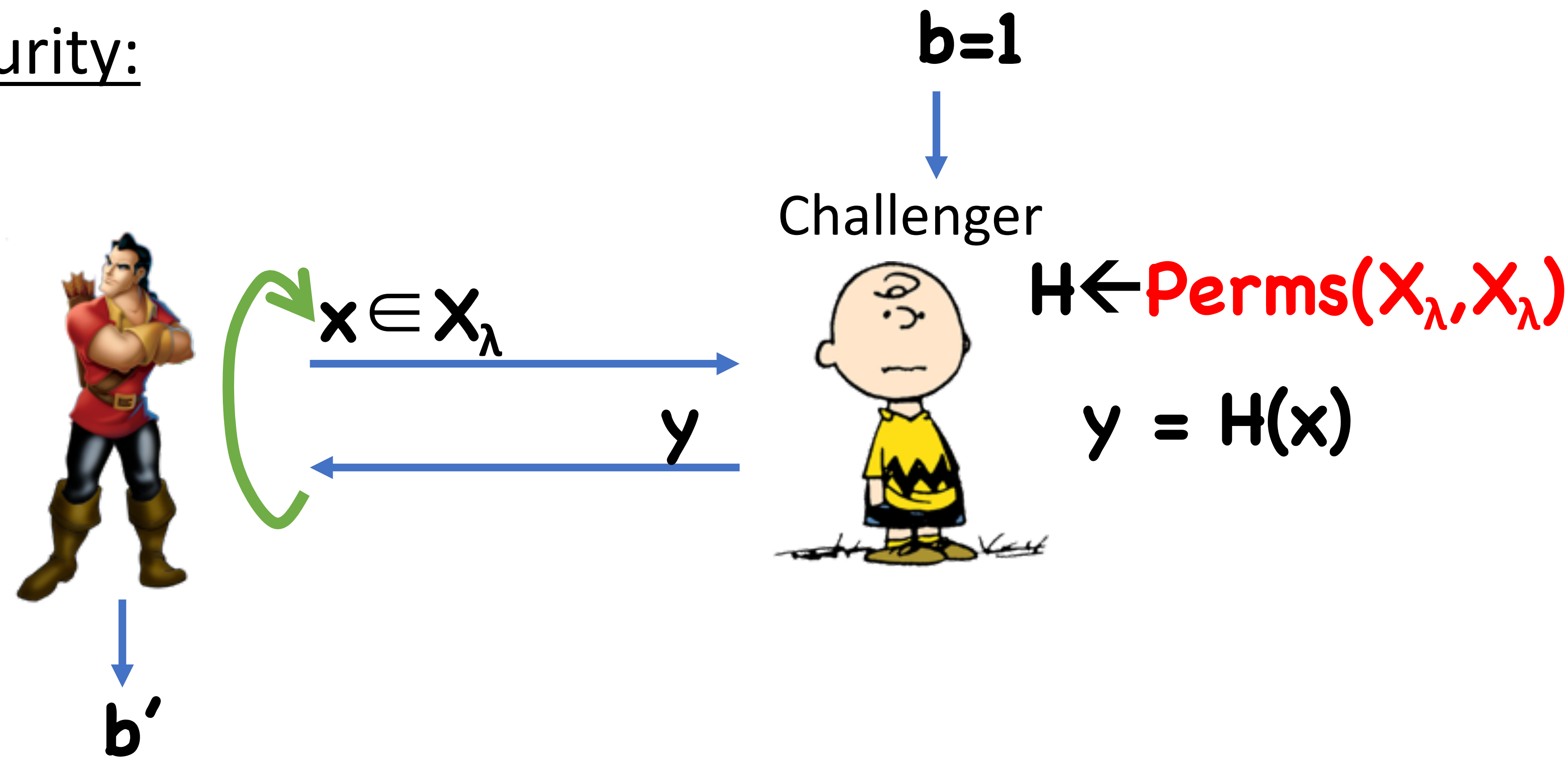


PRF-Exp$_0$(, λ)

Pseudorandom Permutations

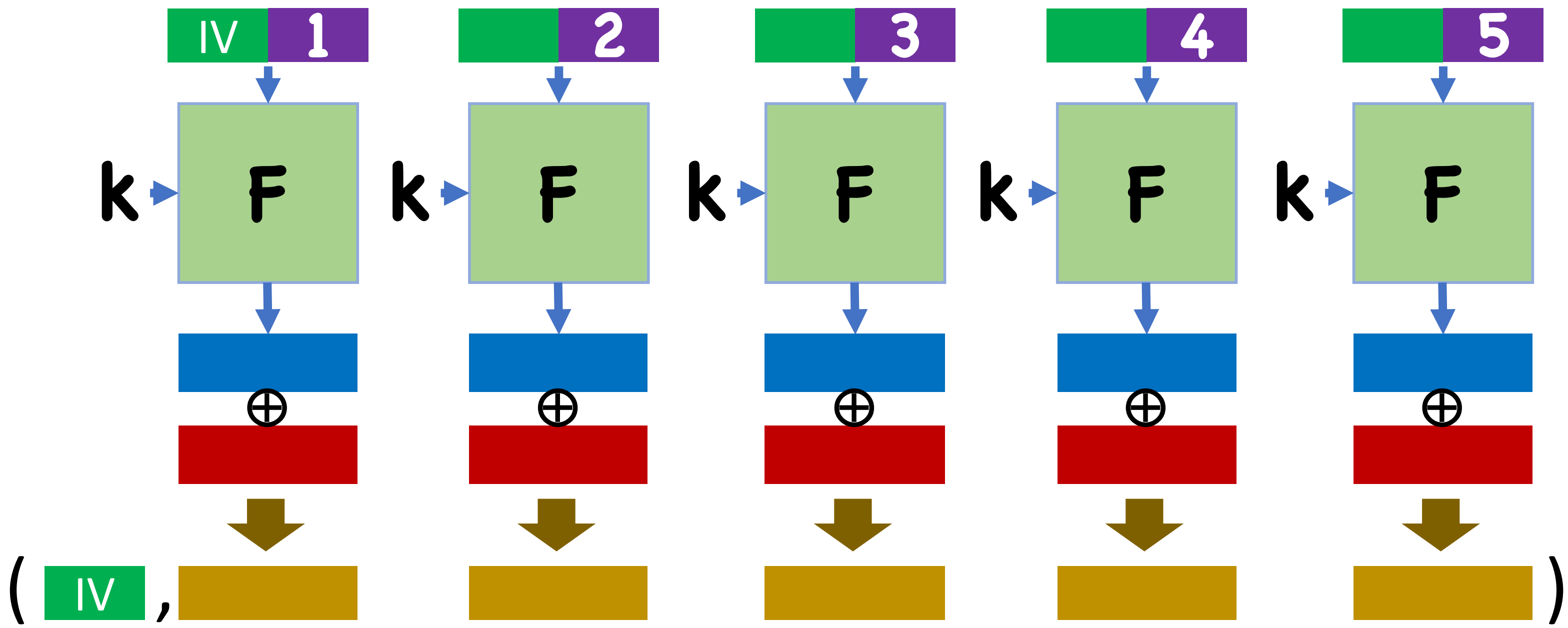
<u>Security:</u>



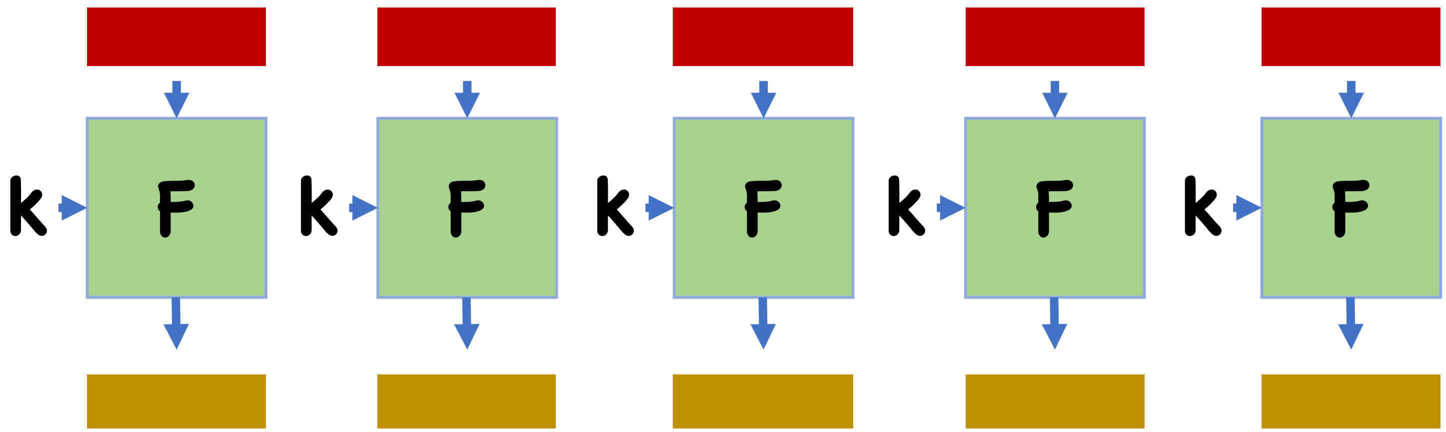
PRF-Exp$_1$( **, λ)**

Theorem: Assuming $|X_\lambda|$ is super-polynomial, a PRP (F, F^{-1}) is secure iff F is secure as a PRF

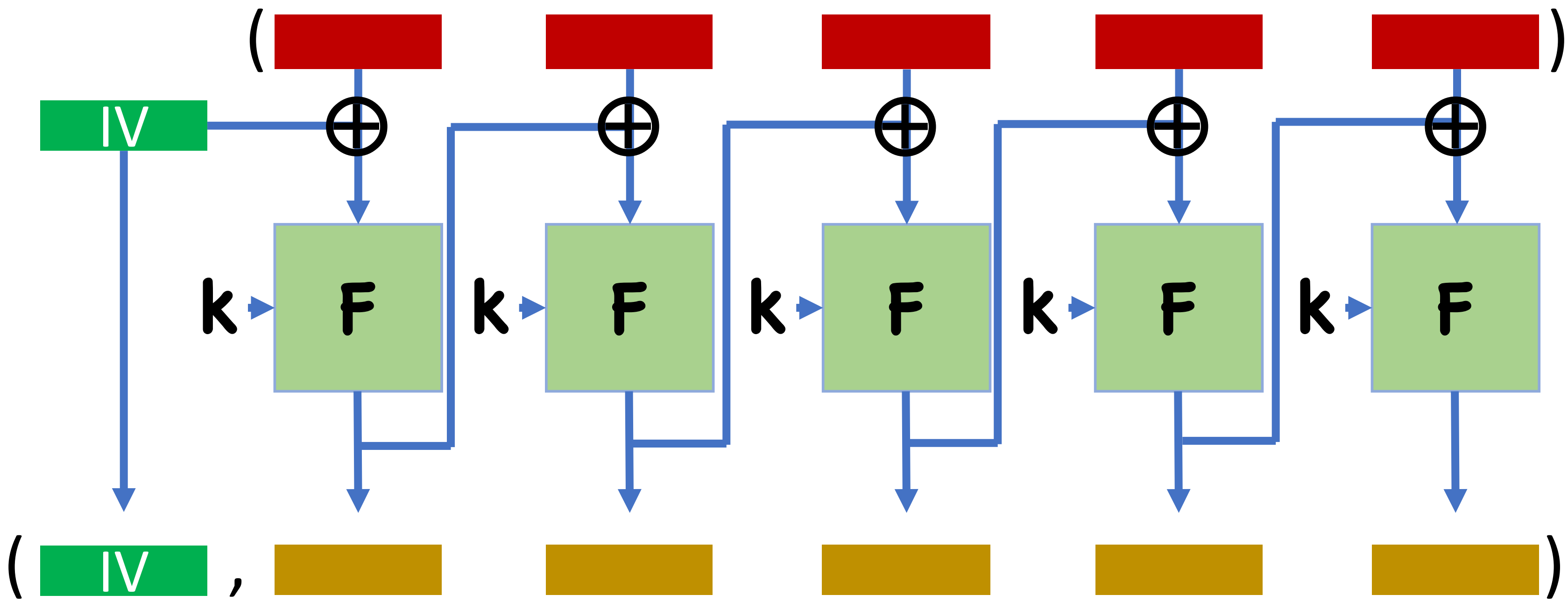
Counter Mode (CTR)



Electronic Code Book (ECB)



Cipher Block Chaining (CBC) Mode



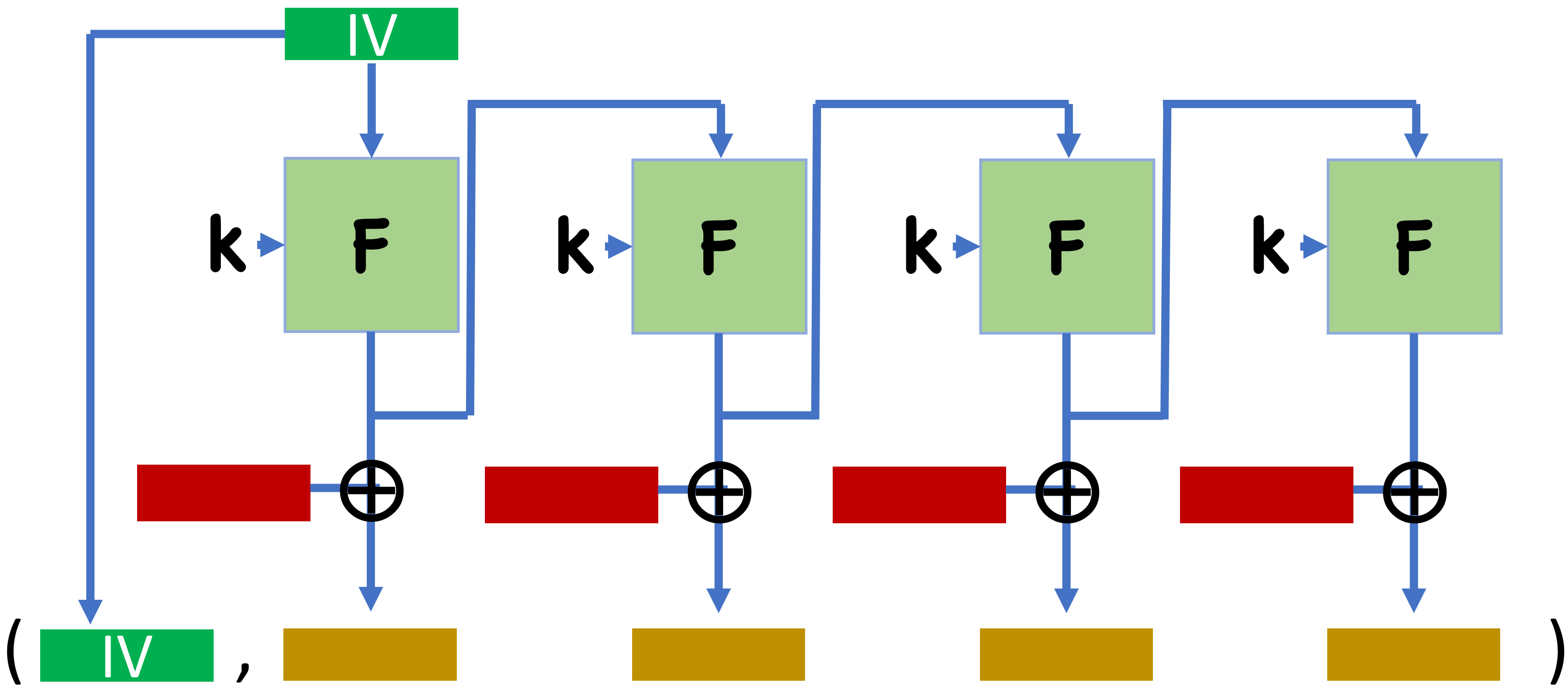
(For now, assume all messages are multiples of the block length)

Today

A few more modes of operation

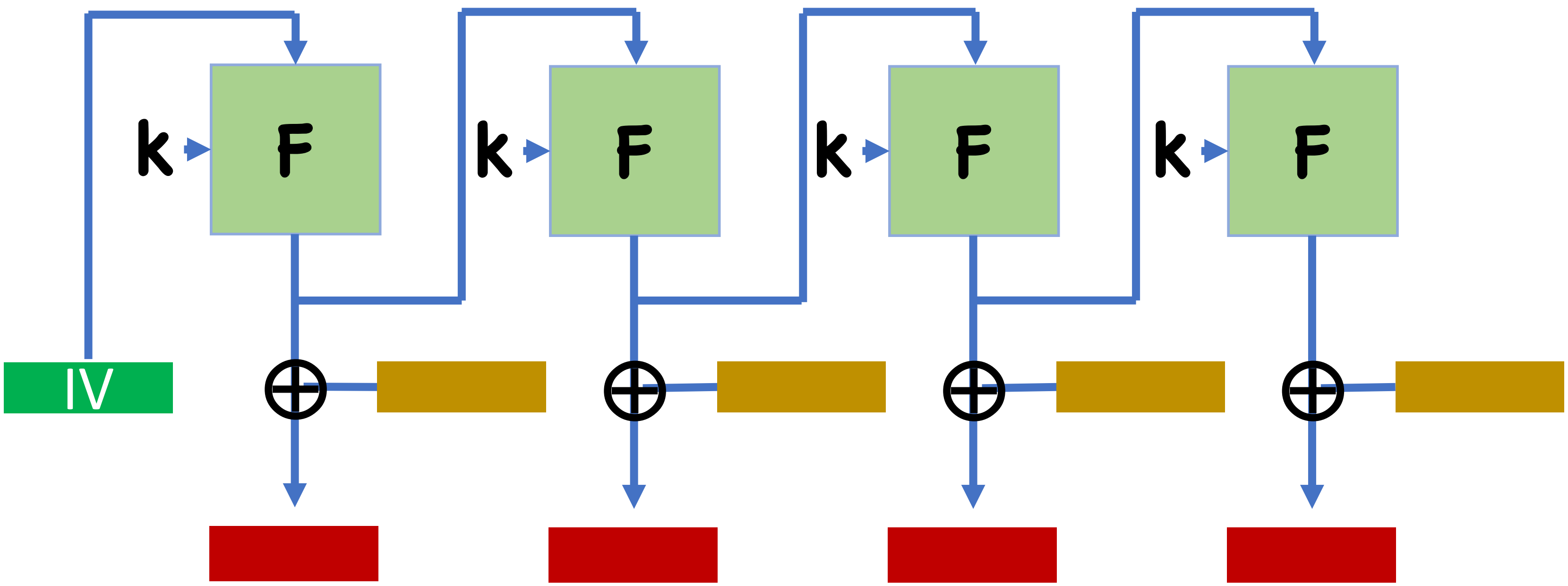
How to construct block ciphers

Output Feedback Mode (OFB)



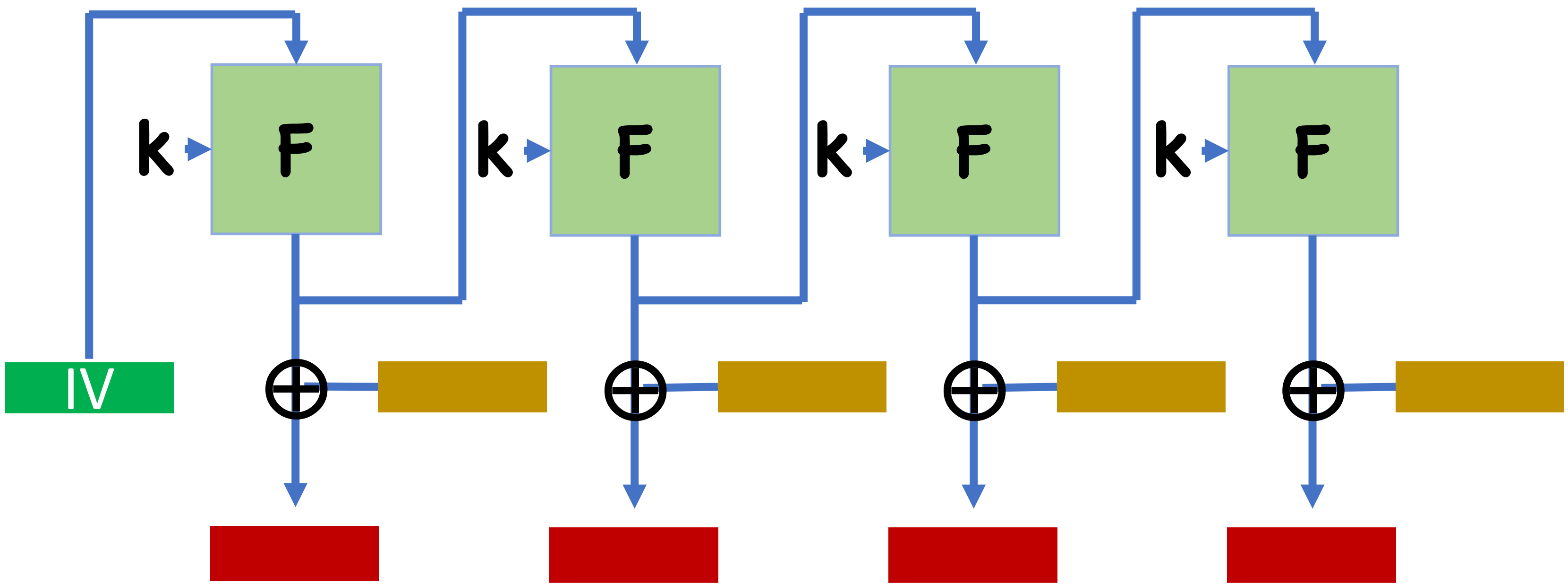
Turn block cipher into stream cipher

OFB Decryption



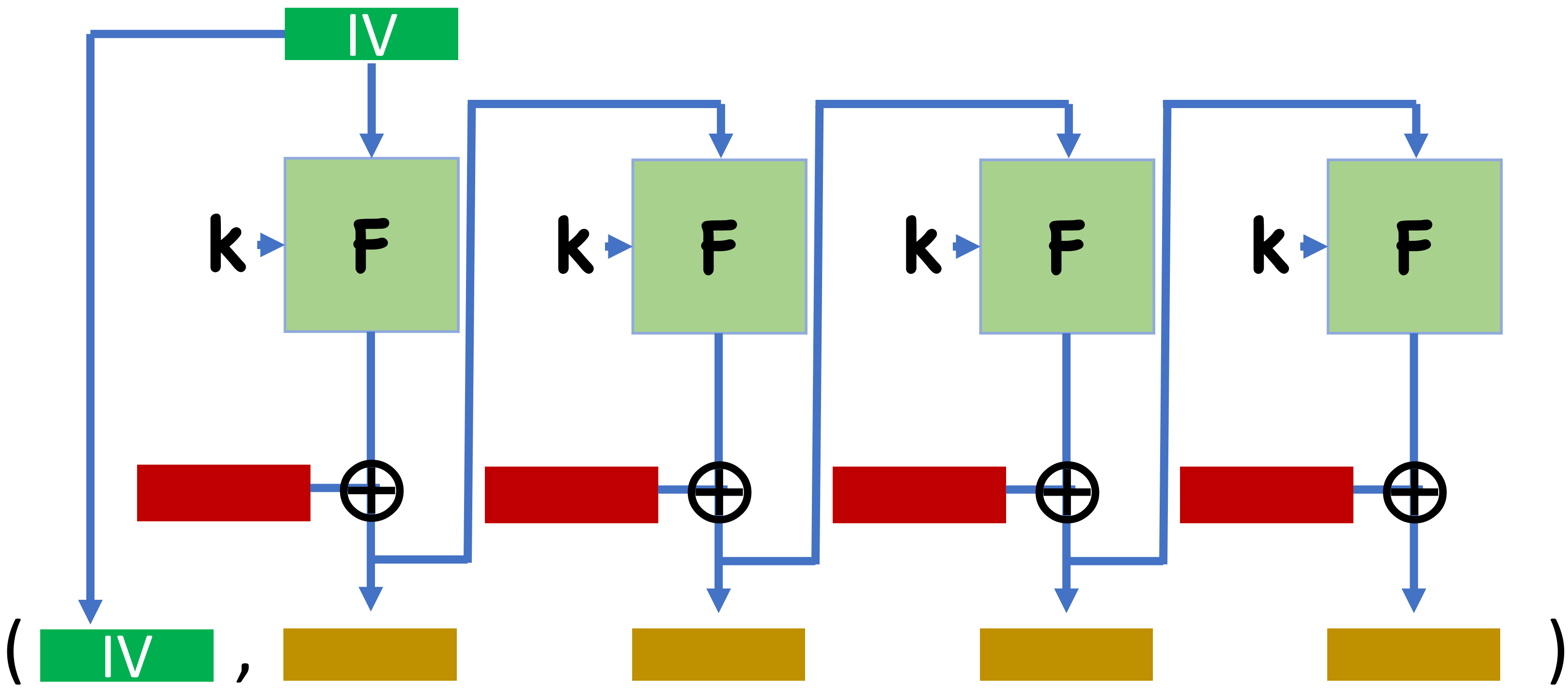
What happens if a block is lost in transmission?

OFB decryption:



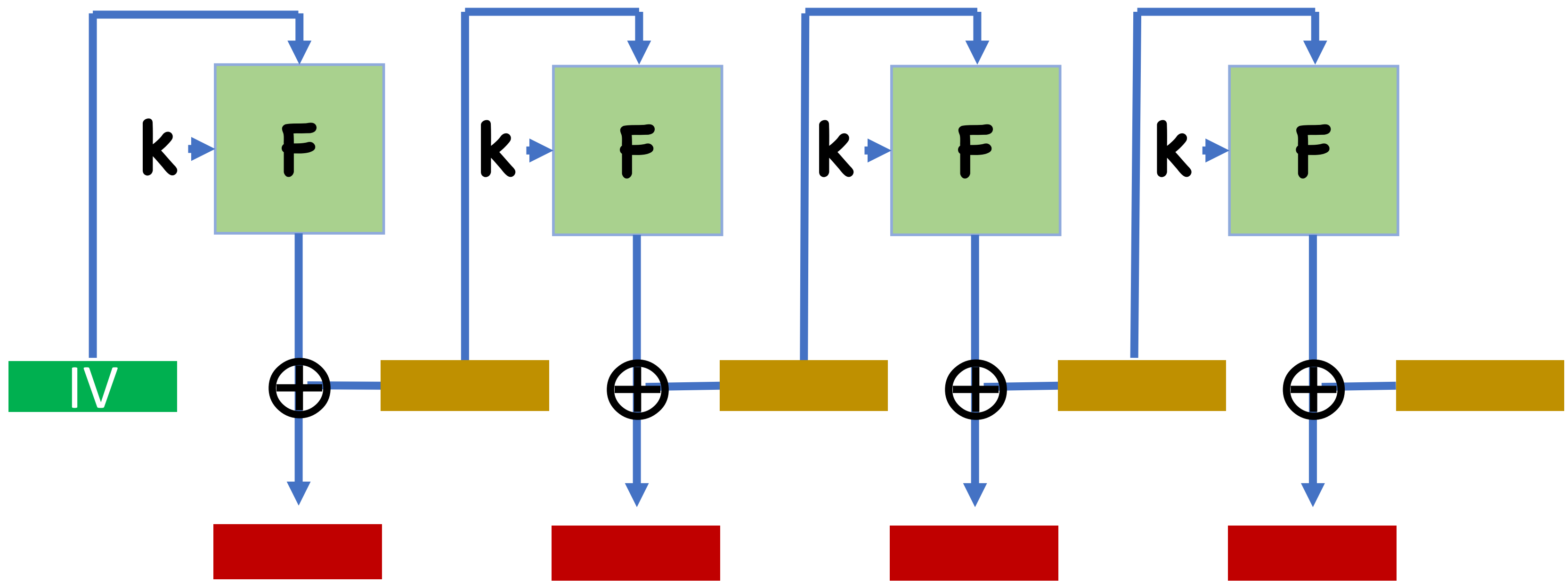
Same goes for CTR mode

Cipher Feedback (CFB)



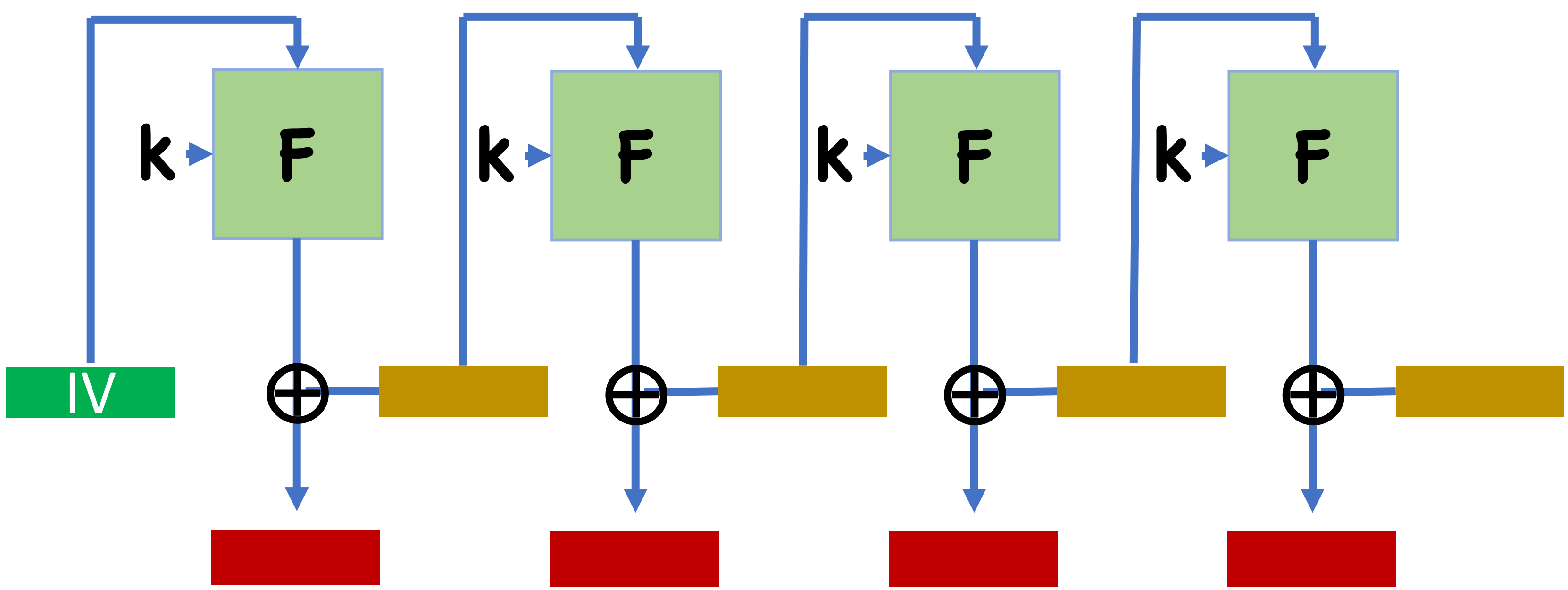
Turn block cipher into **self-synchronizing** stream cipher

CFB Decryption



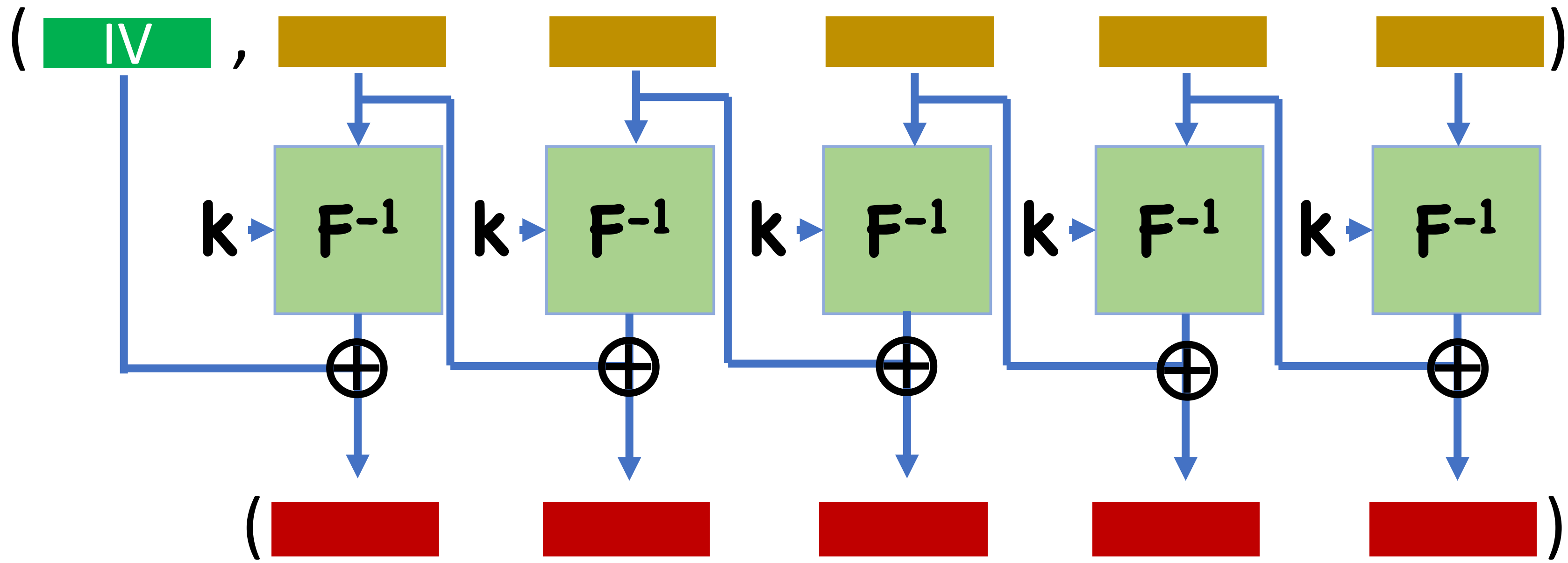
What happens if a block is lost in transmission?

CFB decryption:



What happens if a block is lost in transmission?

What about CBC?



Security of OFB, CFB modes

Security very similar to CBC

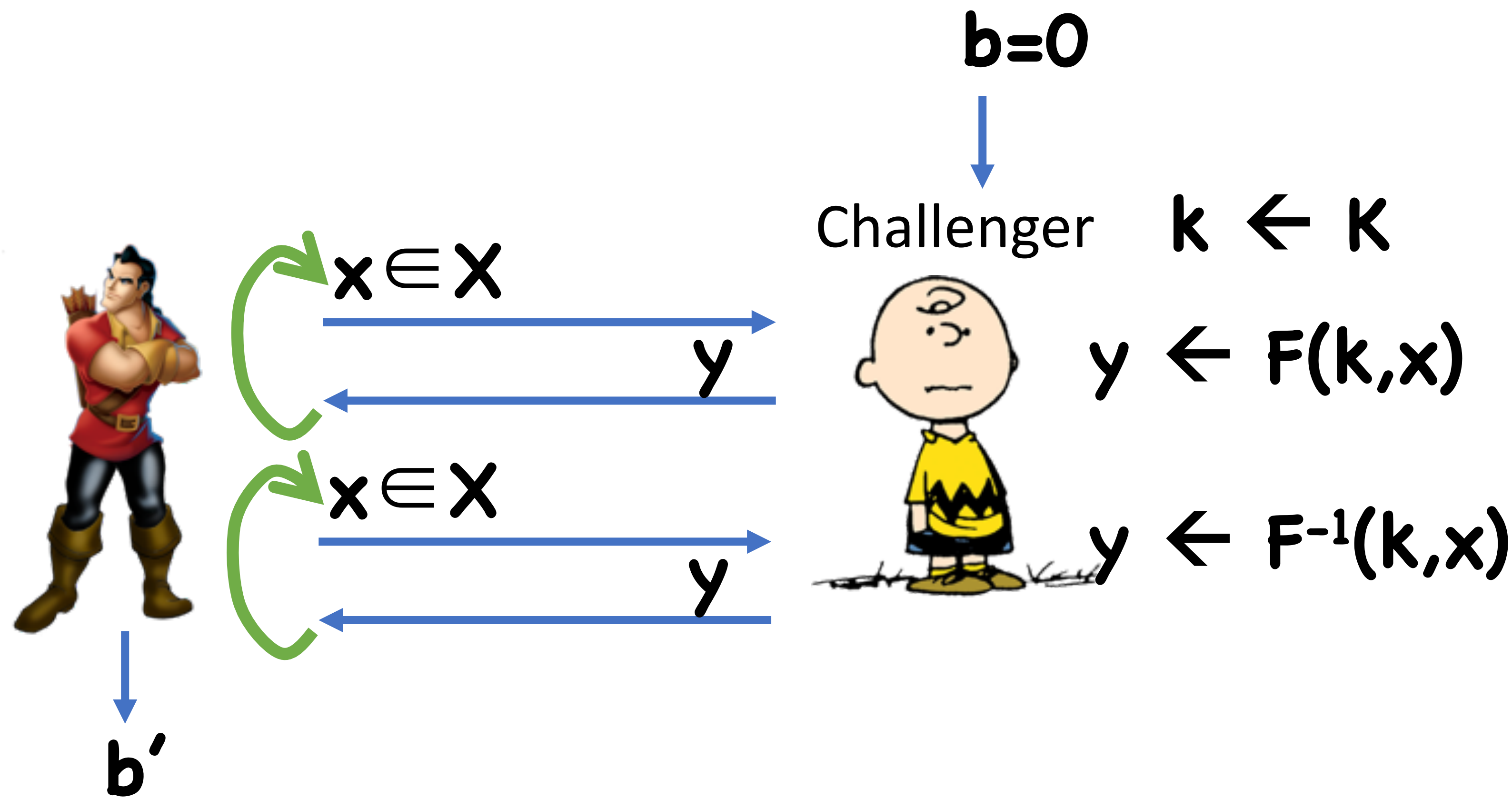
Define 4 hybrids

- 0: encrypt left messages
- 1: replace PRP with random permutation
- 2: encrypt right messages
- 3: replace random permutation with PRP

0,1 and 2,3 are indistinguishable by PRP security

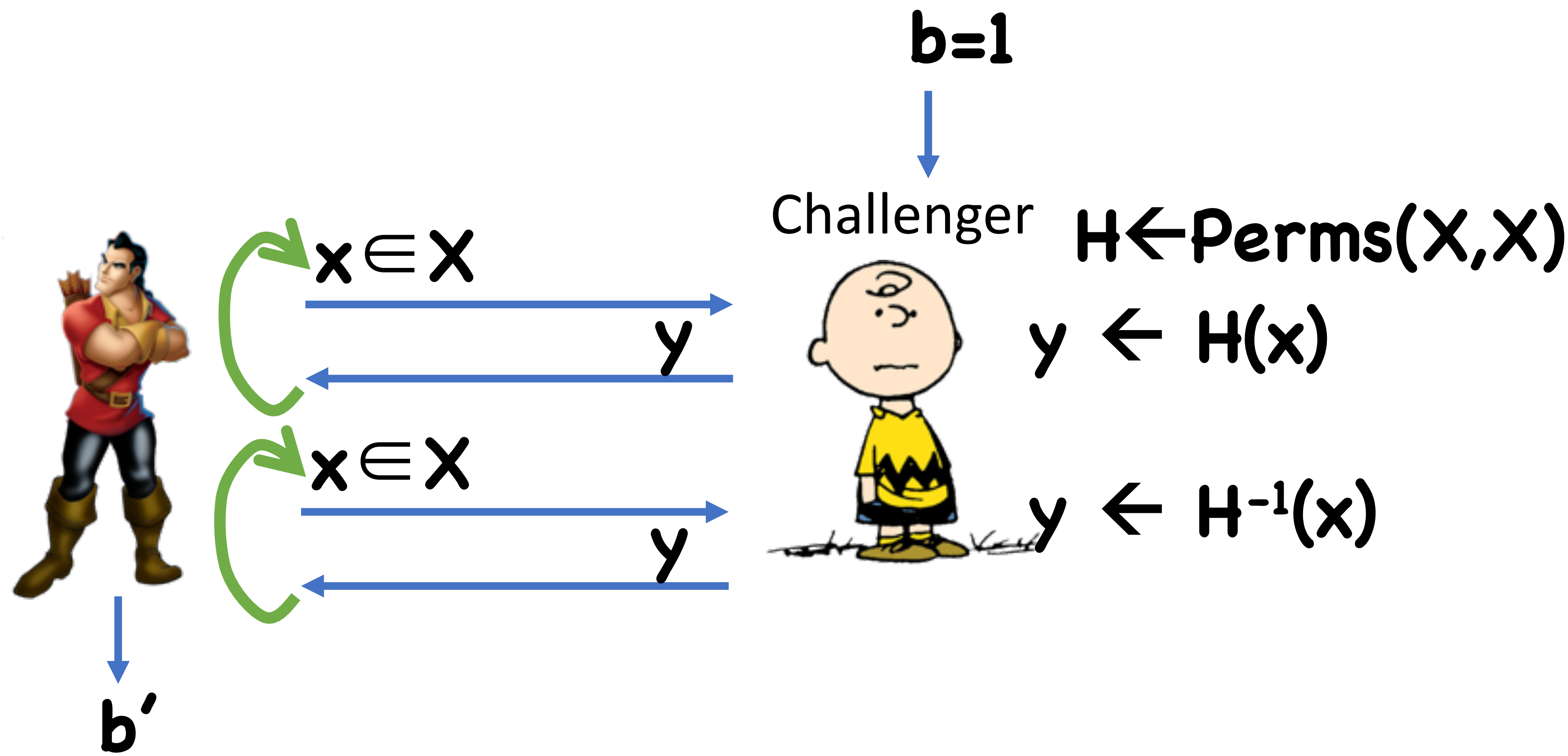
1,2 are indistinguishable since ciphertexts are essentially random

Strong PRPs



PRF-Exp$_0$( **, λ)**

Strong PRPs



$\mathbf{PRF\text{-}Exp_1}($  $, \lambda)$

Theorem: If (F, F^{-1}) is a strong PRP, then so is (F^{-1}, F)

PRPs vs PRFs

In practice, PRPs are the central building block of most crypto

- Also PRFs
- Can build PRGs
- Very versatile

Constructing block ciphers

Difficulties

$\mathbf{2^n!}$ Permutations on $\mathbf{n}$-bit blocks

$\Rightarrow$ $\mathbf{\approx n2^n}$ bits to write down random perm.

Reasonable for very small $\mathbf{n}$ (e.g. $\mathbf{n<20}$), but totally infeasible for large $\mathbf{n}$ (e.g. $\mathbf{n=128}$)

Challenge:

- Design permutations with small description that "behave like" random permutations

Difficulties

For a random permutation **H**, **H(x)** and **H(x')** are (essentially) independent random strings

- Even if **x** and **x'** differ by just a single bit

Therefore, for a random key **k**, changing a single bit of **x** should "affect" all output bits of **F(k,x)**

Definition: For a function $\mathbf{H:\{0,1\}^n \rightarrow \{0,1\}^n}$, we say that bit $\mathbf{i}$ of the input affects bit $\mathbf{j}$ of the output if

For a random $\mathbf{x_1,\ldots,x_{i-1},x_{i+1}, \ldots, x_n}$, if we let
$\mathbf{y=H(x_1\ldots x_{i-1}0x_{i+1}\ldots x_n)}$ and
$\mathbf{z=H(x_1\ldots x_{i-1}1x_{i+1}\ldots x_n)}$
Then $\mathbf{y_j \neq z_j}$ with probability $\mathbf{\approx 1/2}$

Theorem: If $(\mathbf{F},\mathbf{F}^{-1})$ is a secure PRP, then with (with "high" probability over the key $\mathbf{k}$), for the function $\mathbf{F}(\mathbf{k},\bullet)$, every bit of input affects every bit of output

Proof sketch:

- For random permutations this is true
- If bit $\mathbf{i}$ did not affect bit $\mathbf{j}$, we can construct an adversary that distinguishes $\mathbf{F}$ from random

Confusion/Diffusion Paradigm

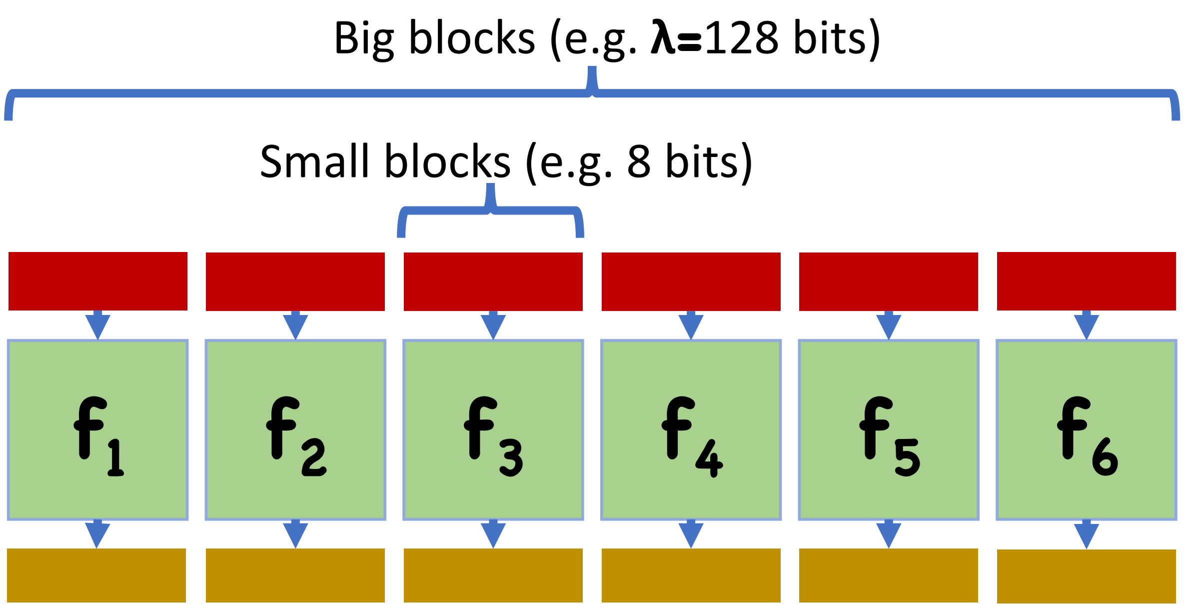
Confusion/Diffusion Paradigm

Goal: build permutation for large blocks from permutations for small blocks

- Small block perms can be made truly random
- Hopefully result is pseudorandom

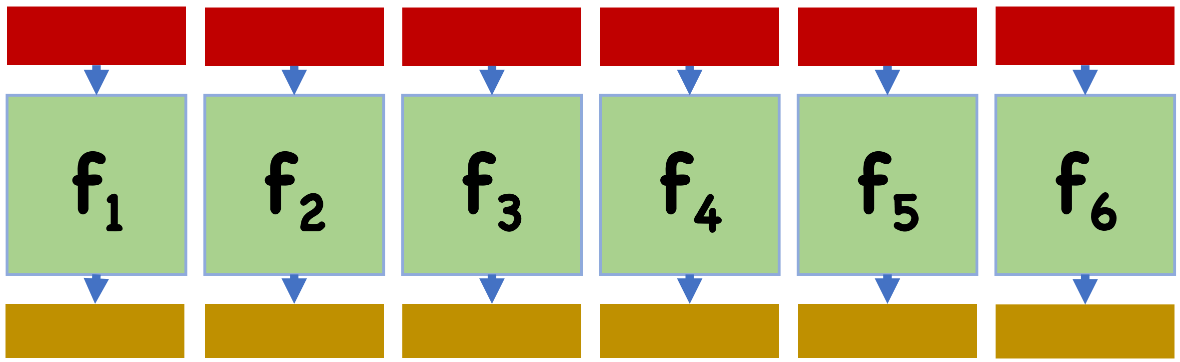
Confusion/Diffusion Paradigm

First attempt: break blocks into smaller blocks, apply smaller permutation blockwise



Key: description of $\mathbf{f}_1$, $\mathbf{f}_2$,…

Confusion/Diffusion Paradigm

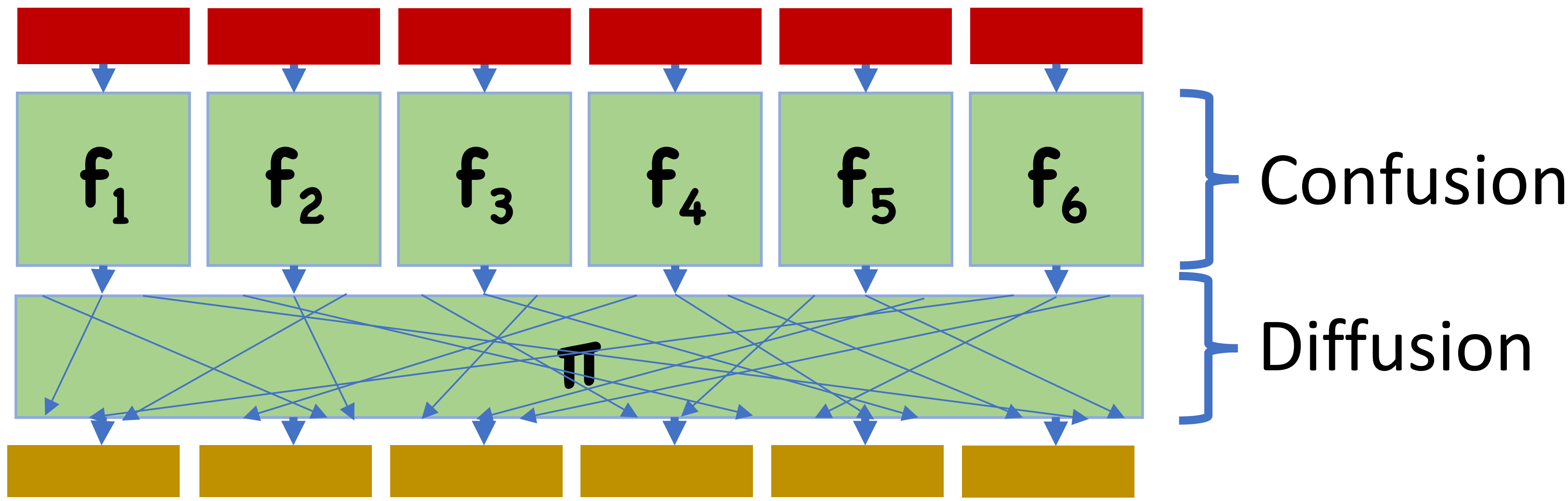


Is this a secure PRP?

- Key size: $\approx(8\times 2^8)\times(\lambda/8) = O(\lambda)$
- Running time: a few table lookups, so efficient
- Security?

Confusion/Diffusion Paradigm

Second attempt: shuffle output bits

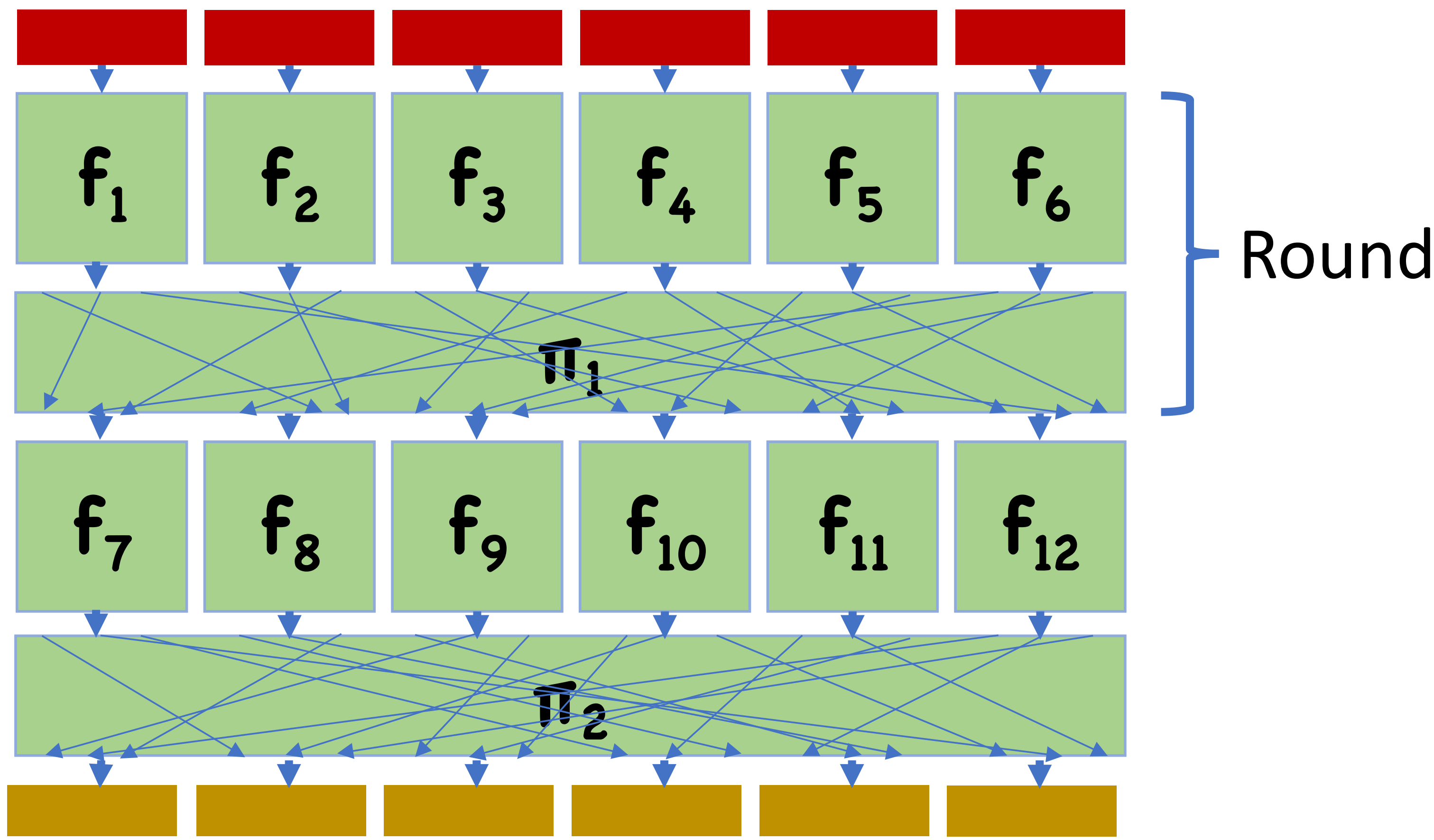


Is this a secure PRP?

- Key size: $\approx 2^8\lambda + \lambda \times \log \lambda$
- Running time: a few table lookups
- Security?

Confusion/Diffusion Paradigm

Third Attempt: Repeat multiple times!



Confusion/Diffusion Paradigm

While single round is insecure, we’ve made progress

- Each bit affects 8 output bits

With repetition, hopefully we will make more and more progress

Confusion/Diffusion Paradigm

With 2 rounds,

- Each bit affects 64 output bits

With 3 rounds, all 128 bits are affected

Repeat a few more times for good measure

Limitations

Describing subs/perms requires many bits

- Key size for $\mathbf{r}$ rounds is approximately $\mathbf{2^8 \times \lambda \times r}$
- Ideally want key size to be **128** (or **256**)

Idea: instead, fix subs/perms

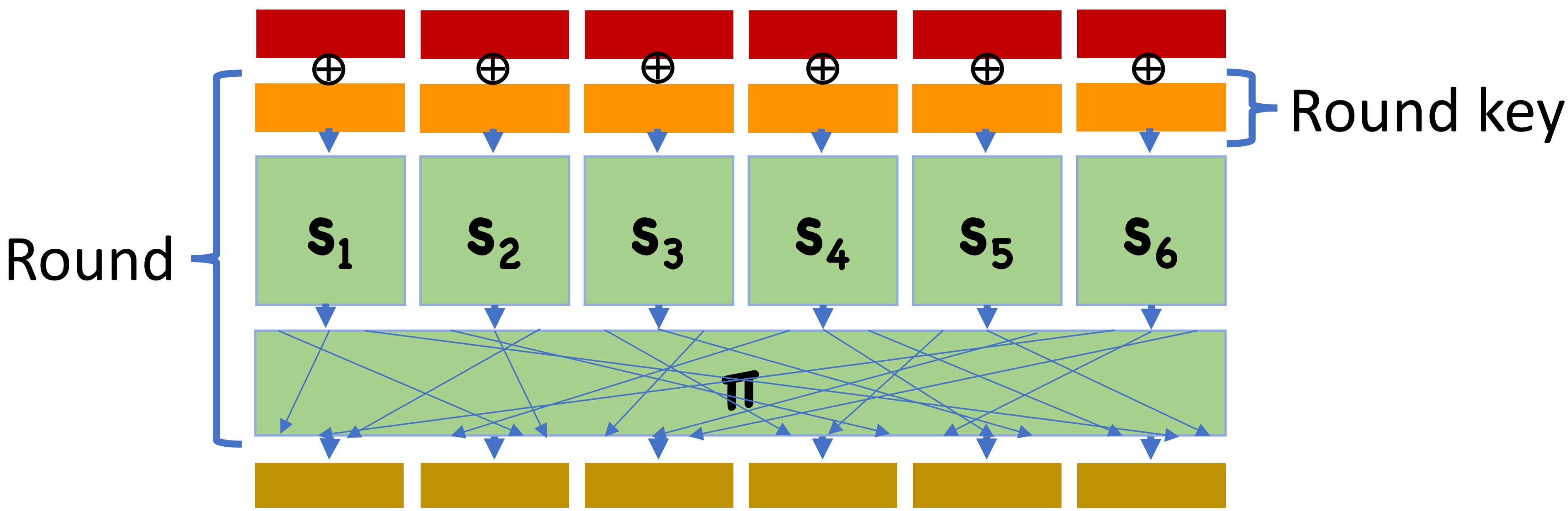
- But then what's the key?

Substitution Permutation Networks

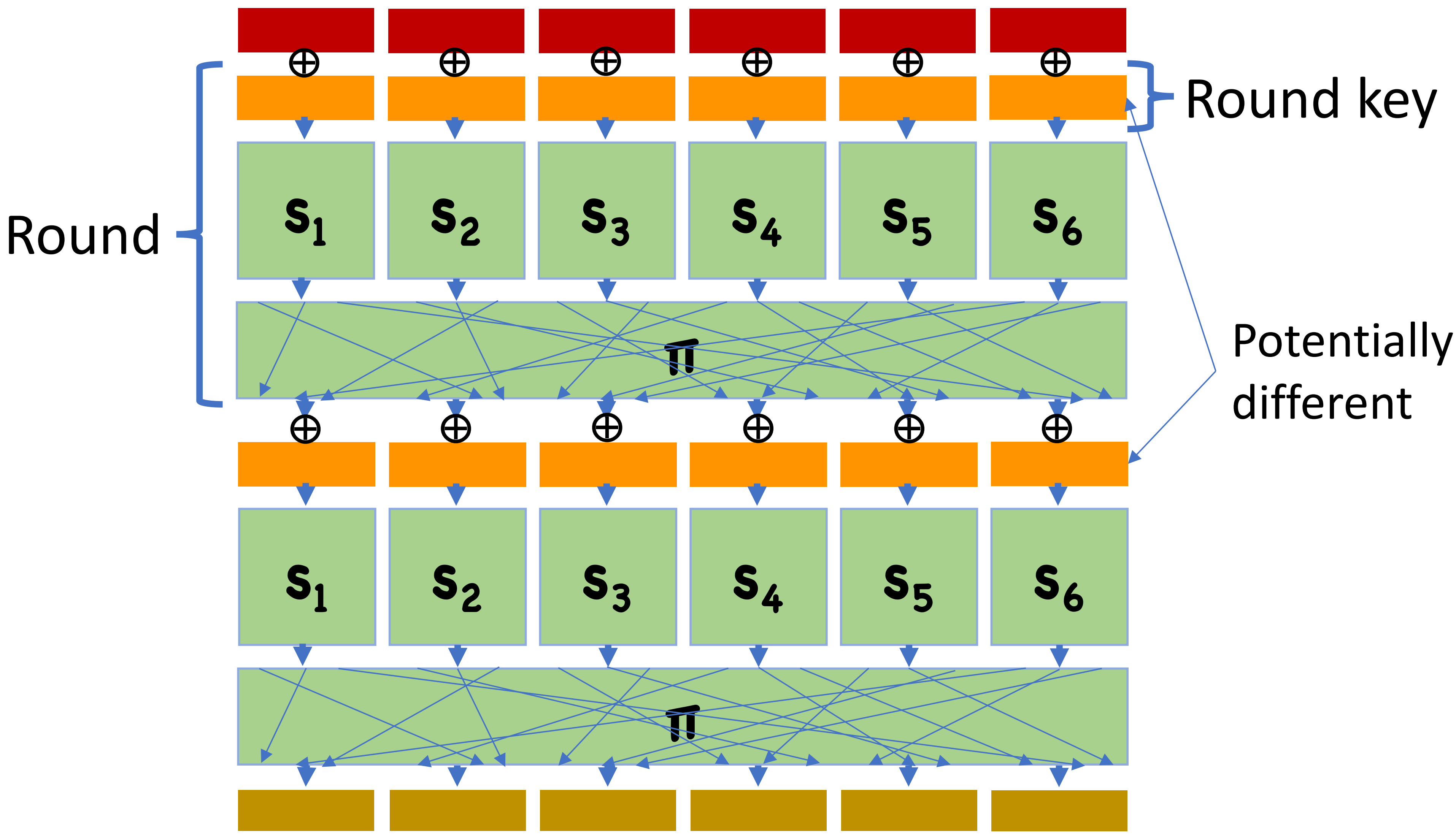
Variant of previous construction

- Fixed public permutations for confusion (called a substitution box, or S-box)
- Fixed public permutation for diffusion (called a permutation box, or P-box)
- XOR “round key” at beginning of each round

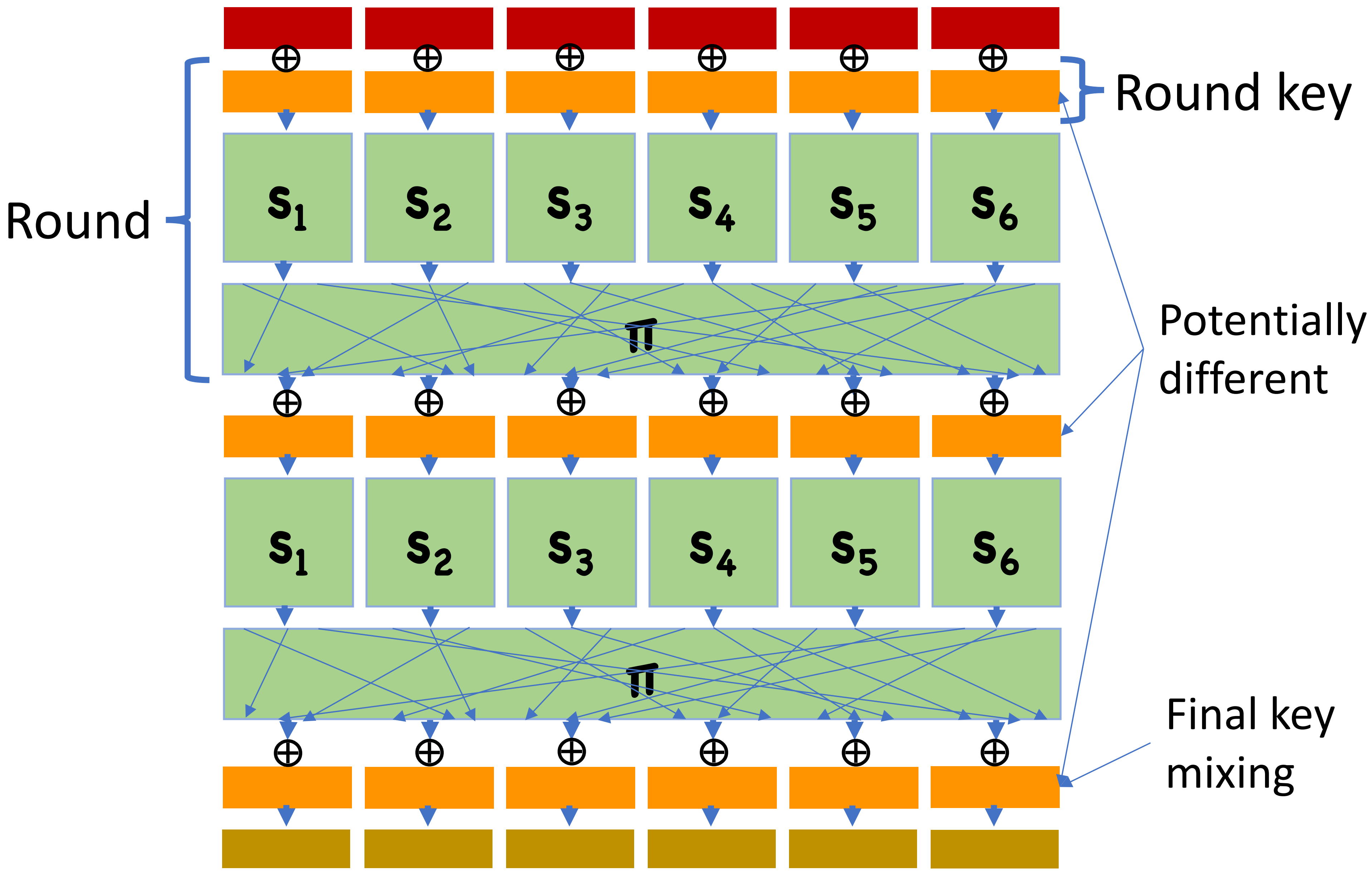
Substitution Permutation Networks



Substitution Permutation Networks



Substitution Permutation Networks



Substitution Permutation Networks

To specify a network, must:

- Specify S-boxes
- Specify P-box
- Specify key schedule (how round keys are derived from master)

Choice of parameters can greatly affect security

Designing SPNs

Avalanche Affect:

- Need S-boxes and mixing permutations to cause every input bit to ”affect” every output bit

One way to guarantee this:

- Changing any bit of S-box input causes at least 2 bits of output to change
- Mixing permutations send outputs of S-boxes into at least 2 different S-boxes for next round
- Sufficiently many rounds are used

Designing SPNs

For strong PRPs, need avalanche in reverse too

- Changing one bit of output of S box changes at least 2 bits of input
- Mixing permutations take inputs for next round from at least two different S-box outputs

Designing S-Boxes

Random?

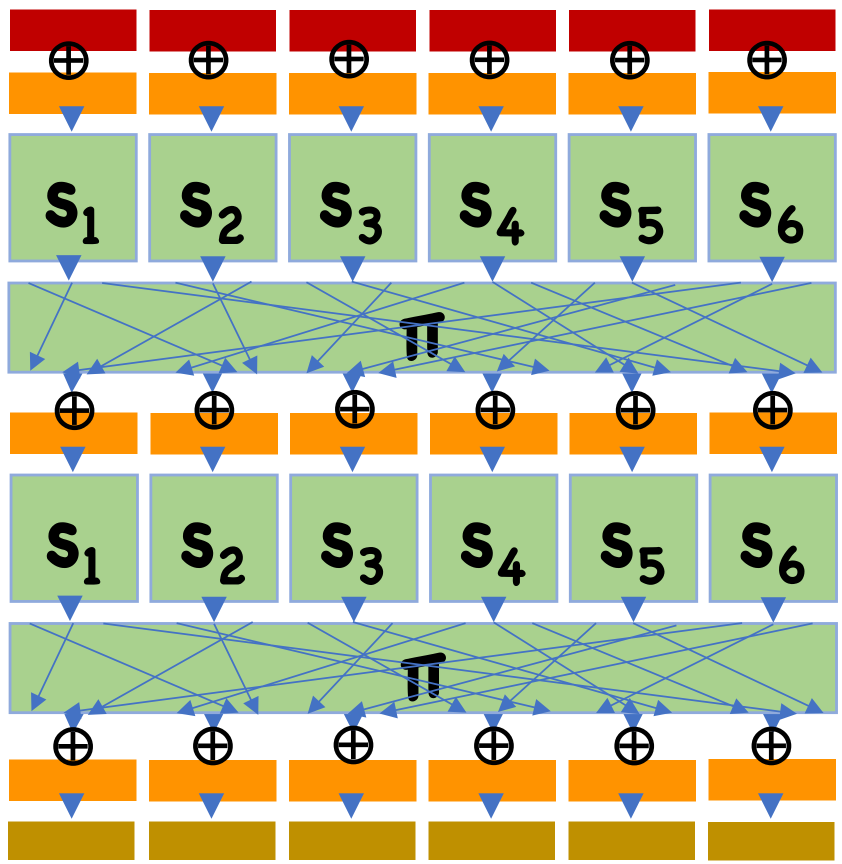
- Let x, x' be two distinct 8-bit values
- $\Pr[S(x)$ and $S(x')$ differ on a single bit$] = 8/255$
- Very high probability that some pair of inputs will have outputs that differ on a single bit

Therefore, must carefully design S-boxes rather than choose at random

Linearity?

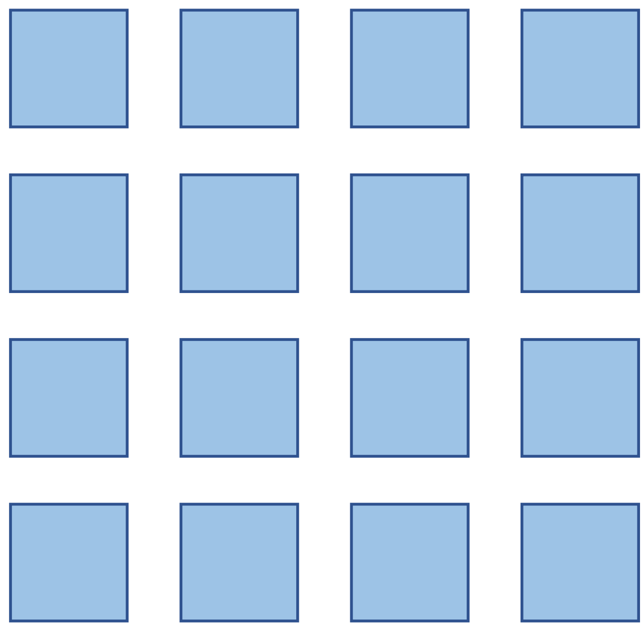
Can S-Boxes be linear?

- That is, $S(x_0) \oplus S(x_1) = S(x_0 \oplus x_1)$?



AES

State = **4×4** grid of bytes



AES

One fixed S-box, applied to each byte

- Step 1: multiplicative inverse over finite field $\mathbb{F}_8$
- Step 2: fixed affine transformation
- Implemented as a simple lookup table

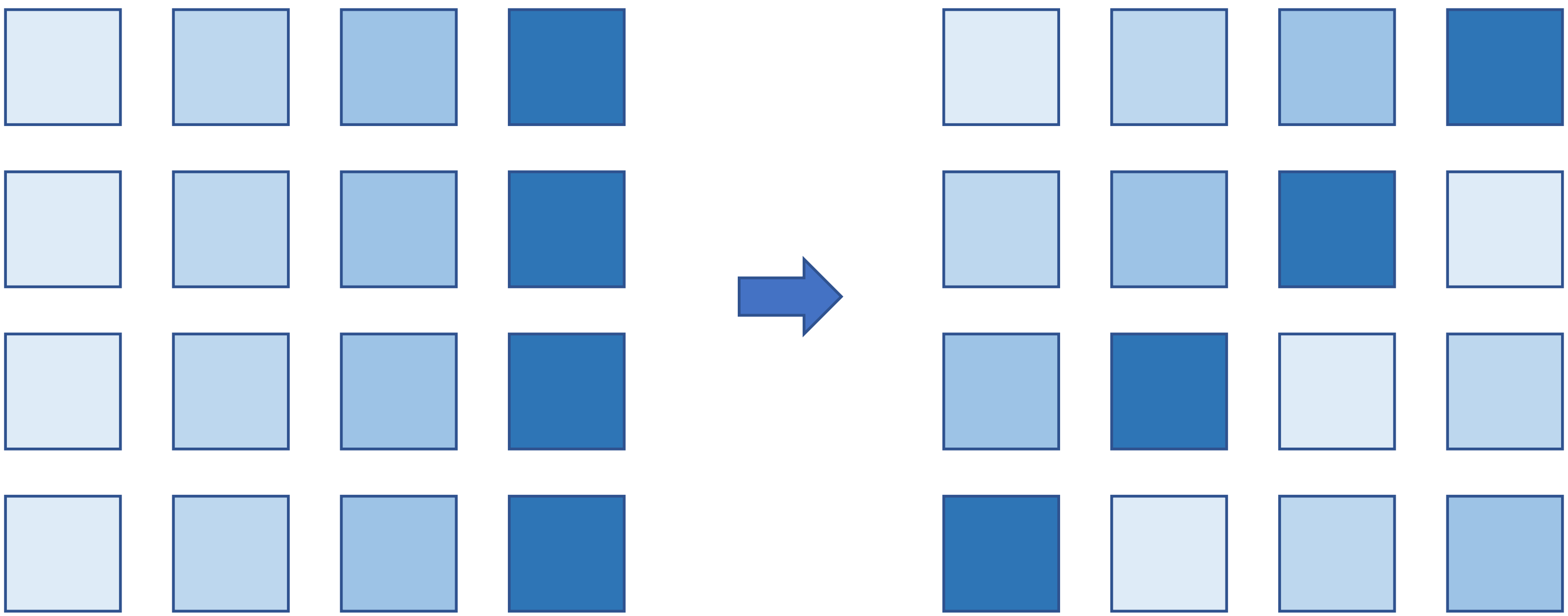
AES

Diffusion (not exactly a P-box):

- Step 1: shift rows
- Step 2: mix columns

AES

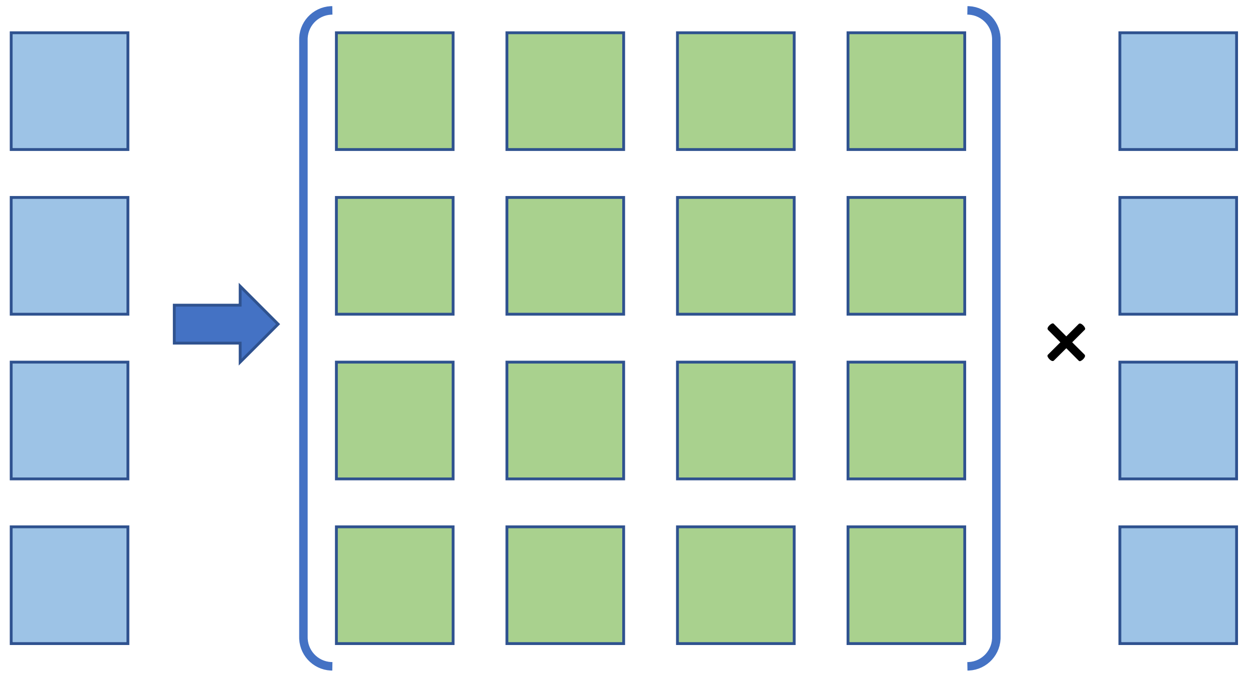
Shift Rows:



AES

Mix Columns

- Each byte interpreted as element of $\mathbb{F}_8$
- Each column is then a length-4 vector
- Apply fixed linear transformation to each column



AES

Number of rounds depends on key size

- 128-bit keys: 10 rounds
- 192-bit keys: 12 rounds
- 256-bit keys: 14 rounds

Key schedule:

- Won’t describe here, but involves more shifting, S-boxes, etc
- Can think of key schedule as a weak PRG

Feistel Networks

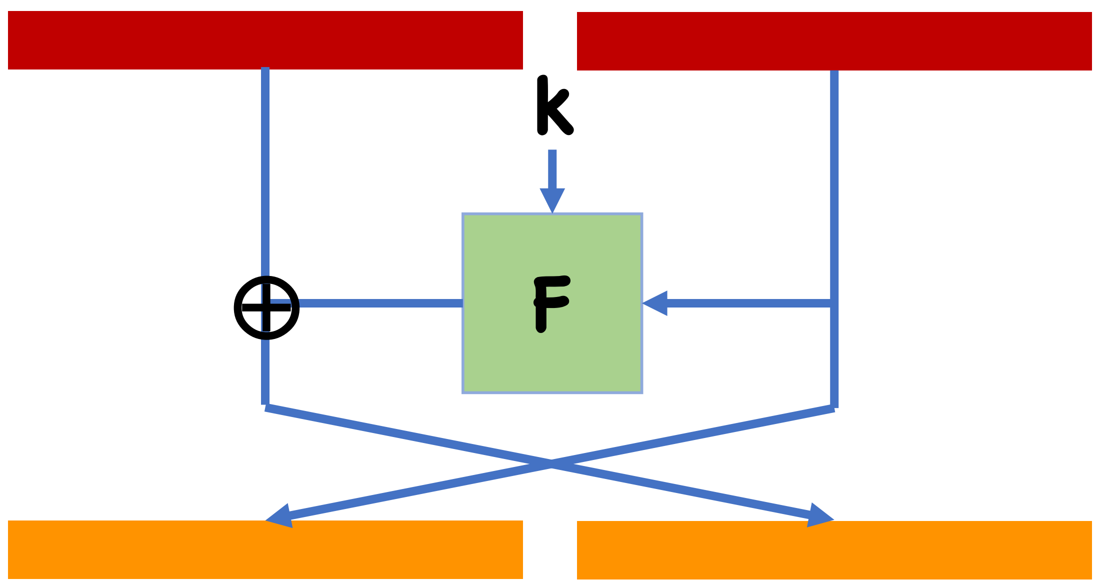
Feistel Networks

Designing permutations with good security properties is hard

What if instead we could built a good permutation from a function with good security properties…

Feistel Network

Convert functions into permutations



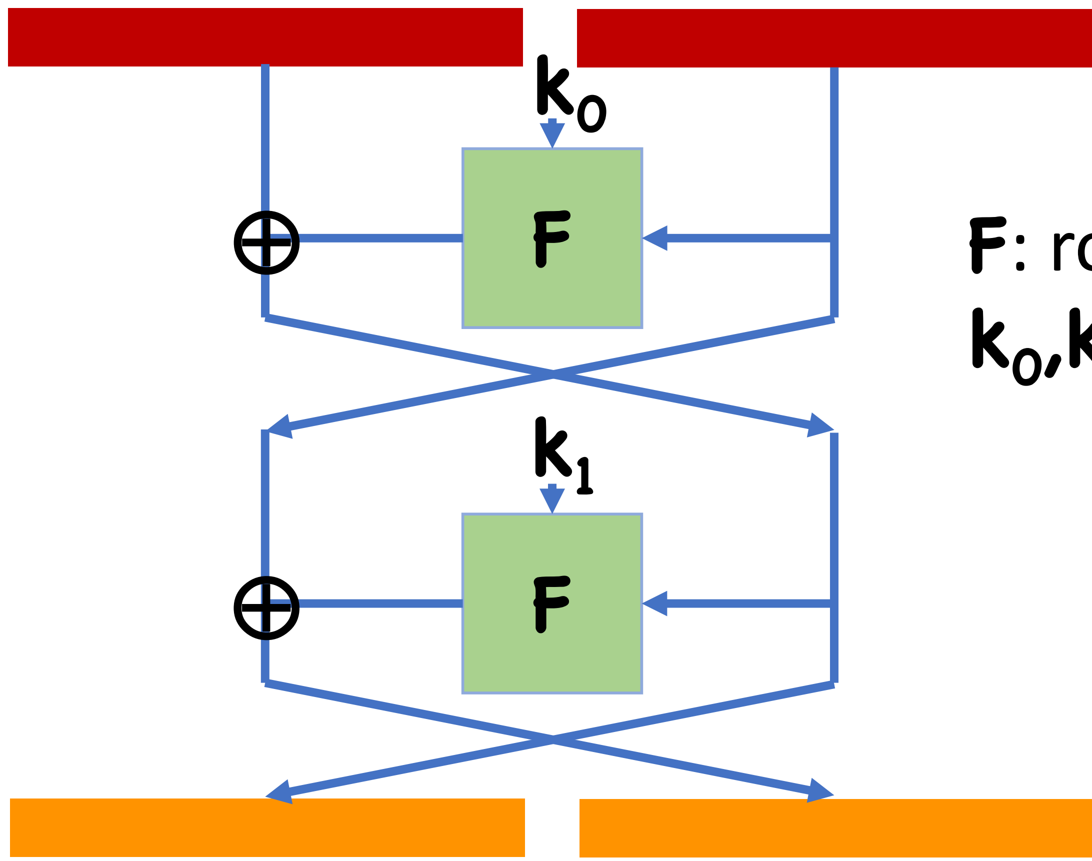
F public

k secret

Can this possibly give a secure PRP?

Feistel Network

Convert functions into permutations



F: round function

$\mathbf{k_0}$,$\mathbf{k_1}$: round keys

Feistel Network

Depending on specifics of round function, different number of rounds may be necessary

- Number of rounds must always be at least 3
- (Need at least 4 for a strong PRP)
- Maybe need even more for weaker round functions

Luby-Rackoff

3- or 4-round Feistel where round function is a PRF

Theorem: If F is a secure PRF, then 3 rounds of Feistel (with independent round keys) give secure PRP. 4 rounds give a strong PRP

- Proof non-trivial, won't be covered in this class

Limitations of Feistel Networks

Turns out Feistel requires block size to be large

- If number of queries ~$2^{\text{block size}/2}$, can attack

Format preserving encryption:

- Encrypted data has same form as original
- E.g. encrypted SSN is an SSN
- Useful for encrypting legacy databases

Sometimes, want a very small block size

Constructing Round Functions

Ideally, ”random looking” functions

Similar ideas to constructing PRPs
- Confusion/diffusion
- SPNs, S-boxes, etc

Key advantage is that we no longer need the functions to be permutations
- S-boxes can be non-permutations

DES

Block size: 64 bits

Key size: 56 bits

Rounds: 16

!!!

DES

Key Schedule:

- Round keys are just 48-bit subsets of master key

Round function:

- Essentially an SPN network

DES S-Boxes

8 different S-boxes, each

- 6-bit input, 4-bit output
- Table lookup: 2 bits specify row, 4 specify column

- Each row contains every possible 4-bit output
- Changing one bit of input changes at least 2 bits of output

DES History

Designed in the 1970's

- At IBM, with the help of the NSA
- At the time, many in academia were suspicious of NSA's involvement
  - Mysterious S-boxes
  - Short key length
- Turns out, S-box probably designed well
  - Resistant to "differential cryptanalysis"
  - Known to IBM and NSA in 1970's, but kept secret
- Essentially only weakness is the short key length
  - Maybe secure in the 1970's, definitely not today

DES Security Today

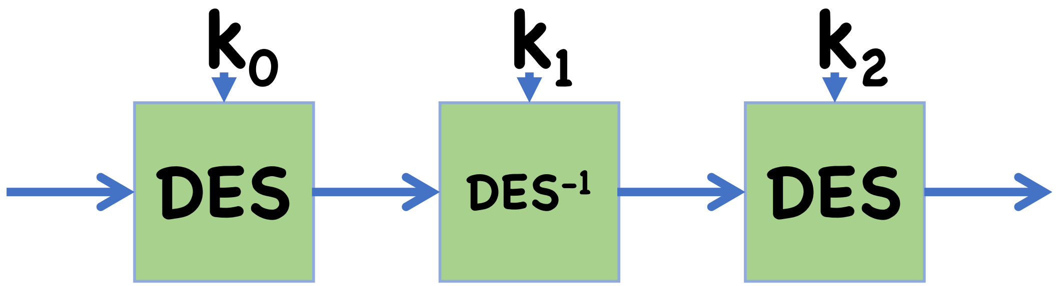
Seems like a good cipher, except for its key length and block size

What's wrong with a small block size?

- Remember for e.g. CTR mode, IV is one block
- If two identical IV's seen, attack possible
- After seeing q ciphertext, probability of repeat IV is roughly $\mathbf{q^2/2}^{\text{block length}}$
- Attack after seeing ≈ billion messages

3DES: Increasing Key Length

3DES key = Apply DES three times with different keys



Why three times?

- Later: “meet in the middle attack” renders 2DES no more secure than 3DES

Why inverted second permutation?

Attacks on block ciphers

Brute Force Attacks

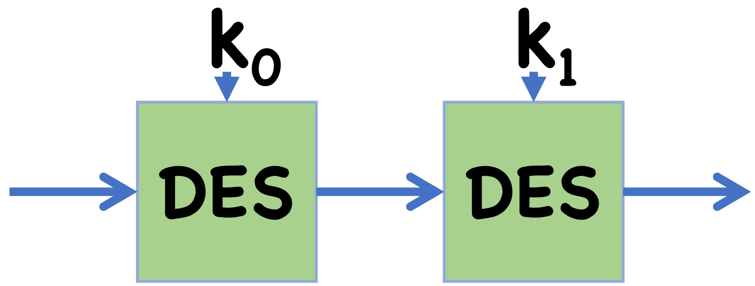
Suppose attacker is given a few input/output pairs

Likely only one key could be consistent with this input/output

Brute force search: try every key in the key space, and check for consistency

Attack time: $2^{\text{key length}}$

Insecurity of 2DES

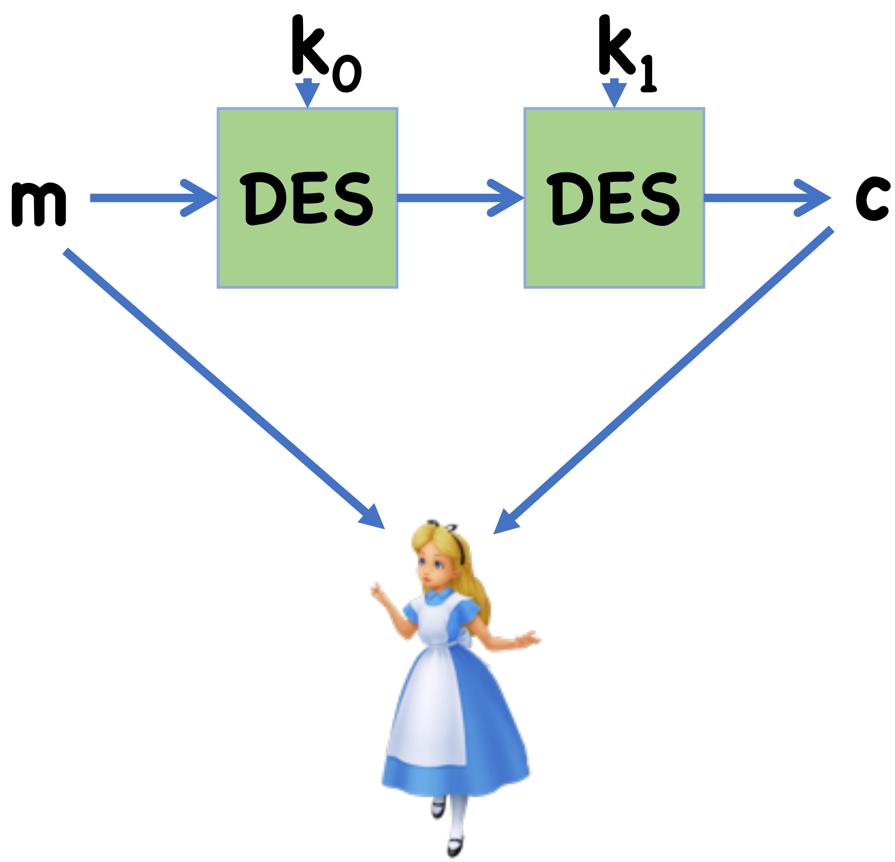


DES key length: 56 bits
2DES key length: 112 bits
Brute force attack running time: 2^{112}

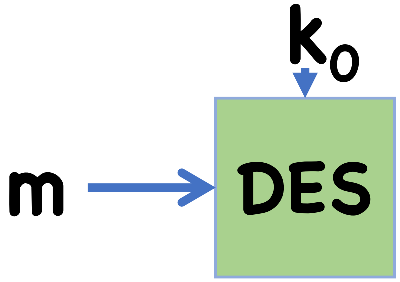
Meet In The Middle Attacks

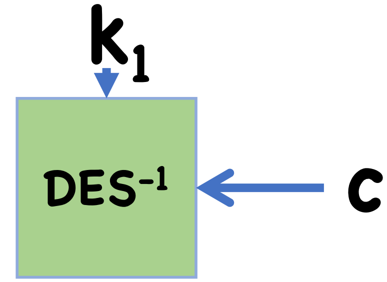
For 2DES, can actually find key in 2^{56} time

- Also $\approx 2^{56}$ space



Meet In The Middle Attacks





k_0	$d = DES(k_0,m)$
0	52
1	93
2	03
3	96
4	20
5	49
…	…

k_1	$d = DES^{-1}(k_1,m)$
0	69
1	10
2	86
3	49
4	99
5	08
…	…

Meet In The Middle Attacks

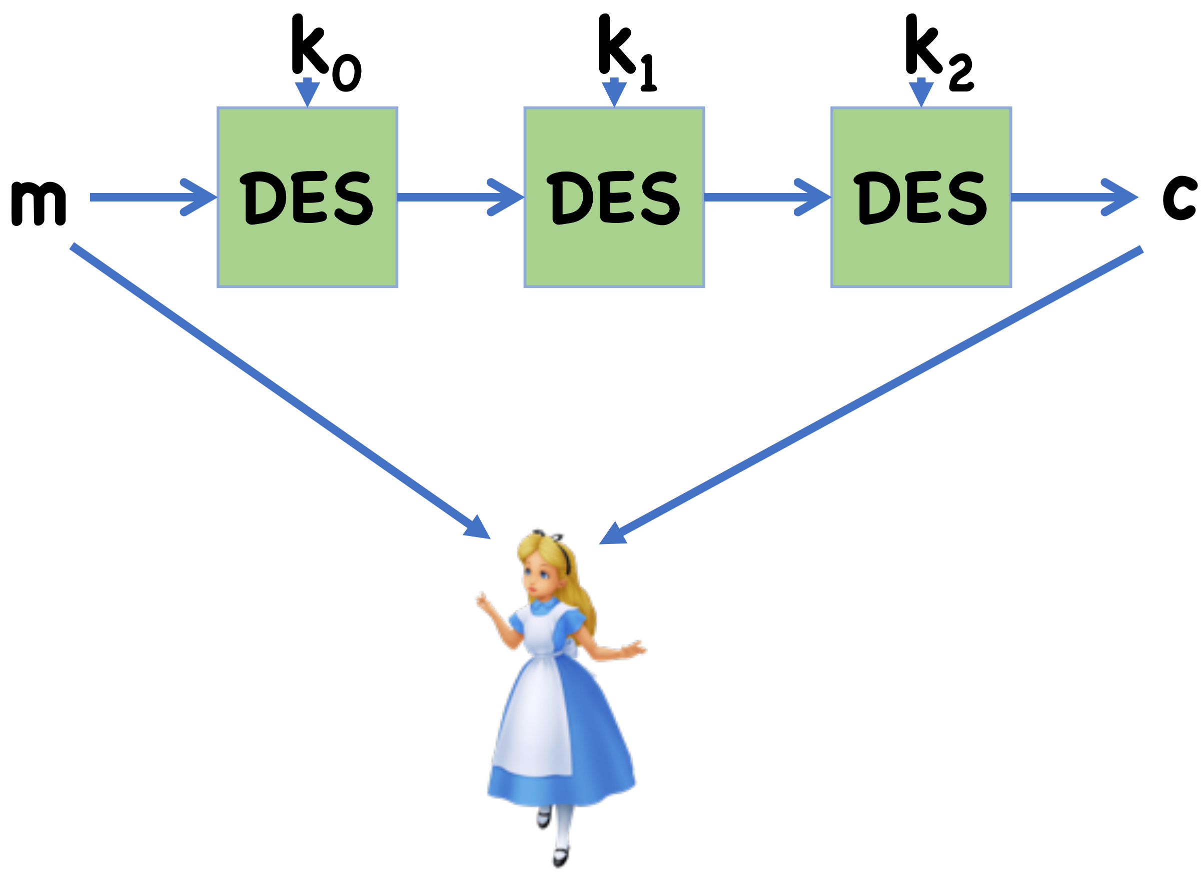
Complexity of meet in the middle attack:

- Computing two tables: time, space $2 \times 2^{\text{key length}}$
- Slight optimization: don't need to actually store second table

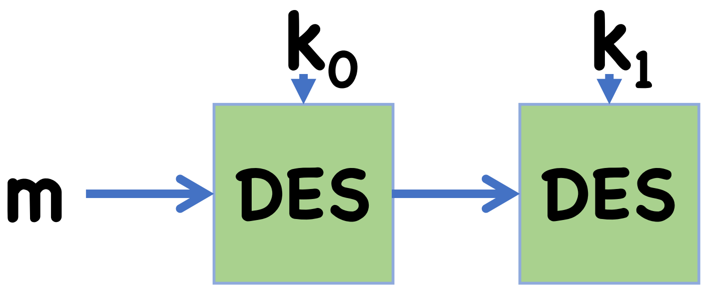
On 2DES, roughly same time complexity as brute force on DES

MITM Attacks on 3DES

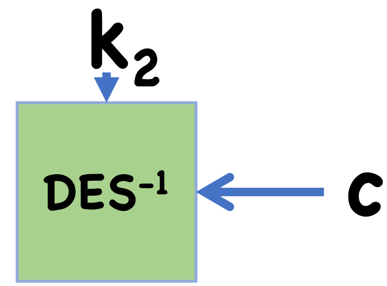
MITM attacks also apply to 3DES…



MITM for 3DES



m,c



k_0	k_1	$d = DES(k_0,m)$
0	0	52
0	1	93
...	...	03
5	6	96
5	7	20
5	8	49
...		...

k_2	$d = DES^{-1}(k_2,m)$
0	69
1	10
2	86
3	49
4	99
5	08
...	...

MITM for 3DES

No matter where "middle" is, need to have two keys on one side

- Must go over 2^{112} different keys

Space?

While 3DES has 168 bit keys, effective security is 112 bits

Generalizing MITM

In general, given $\mathbf{r}$ rounds of a block cipher with $\mathbf{t}$-bit keys,

- Attack time: $\mathbf{2}^{t\lceil r/2 \rceil}$
- Attack space: $\mathbf{2}^{t\lceil r/2 \rceil}$

Brute Force vs. Generic Attacks

MITM attacks on iterated block ciphers are *generic*

- Attack exists independent of implementation details of block cipher

However, still beats a *brute force*

- Doesn’t simply try every key

Reminders

HW3 Due March 5th

PR1 Due **March 12th**

- No late days