COS433/Math 473: Cryptography

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Previously on COS 433…

Perfect Security for Multiple Messages

Definition: A stateless scheme **(Enc,Dec)** has **perfect secrecy for n messages** if, for any two sequences of messages $(m_0^{(i)})_{i\in[d]}$, $(m_1^{(i)})_{i\in[d]} \in M^d$

$$\left(Enc(K,\ m_0^{(i)}\ )\right)_{i\in[d]} \stackrel{d}{=} \left(Enc(K,\ m_1^{(i)}\ )\right)_{i\in[d]}$$

Notation: $(\ f(i)\ )_{i\in[d]} = (\ f(1),\ f(2),\ \ldots,\ f(n)\ )$

Randomized Encryption

Syntax:

- Key space $\mathbf{K}$ (usually $\{0,1\}^\lambda$)
- Message space $\mathbf{M}$ (usually $\{0,1\}^n$)
- Ciphertext space $\mathbf{C}$ (usually $\{0,1\}^m$)
- **Enc:** $K \times M \rightarrow C$ (potentially probabilistic)
- **Dec:** $K \times C \rightarrow M$ (usually deterministic)

Correctness:

- For all $k \in K$, $m \in M$,

$$\Pr[\ \mathrm{Dec}(k, \mathrm{Enc}(k,m)\ ) = m\ ] \quad = \quad 1$$

Theorem: No stateless *randomized* encryption scheme can have perfect security for multiple messages

Today: Relaxing Perfect Secrecy

What do we do now?

Tolerate tiny probability of distinguishing

- If $\Pr[c^{(0)} = c^{(1)}] = 2^{-128}$, in reality never going to happen

How Small Is Ok?

Practice:

- Something unlikely to happen in lifetime of data/person/civilization/universe

- Typically something like 2^{-80}, 2^{-128}, or maybe 2^{-258}
  - Being struck by lightning twice: 2^{-23}
  - Winning the lottery: 2^{-26}
  - World-ending asteroid while on this slide: 2^{-46}

How Small Is Ok?

Theory:

- Maybe things will change as technology improves
- Want a more conceptual answer
- Absolute constants unsatisfactory
- Instead, use ``negligible’’ functions

Negligible functions

Def: A function $\mathbf{f}$ is **polynomial** if $\mathbf{f(n) = O(n^c)}$ for some constant $\mathbf{c}$

Def: A function $\mathbf{g}$ is **super-polynomial** if, for every polynomial $\mathbf{f}$, $\mathbf{f(n) = O(g(n))}$

Def: A function $\mathbf{p}$ is **inverse polynomial** if $\mathbf{1/p(n)}$ is polynomial

Def: A function $\boldsymbol{\varepsilon}$ is **negligible** if, for every inverse polynomial $\mathbf{p}$, $\boldsymbol{\varepsilon}\mathbf{(n) = O(p(n))}$

(equivalently, $\mathbf{1/}\boldsymbol{\varepsilon}$ is super-polynomial)

Examples

2^n	super-polynomial
$n^{-n/7}$	negligible
$3^{-5\log n}$	inverse polynomial
$1.5^{-\sqrt[3]{n}}$	negligible
$8^{\log^3 n}$	super-polynomial
$(\log n)/n$	inverse polynomial

Security Parameter $\boldsymbol{\lambda}$

Additional input to system, dictates "security level"

Key, message, ciphertext size all **polynomial** in $\boldsymbol{\lambda}$

Probability of adversary success is **negligible** in $\boldsymbol{\lambda}$

Defining Encryption Again

Syntax:

- Key space $\mathbf{K_\lambda}$
- Message space $\mathbf{M_\lambda}$
- Ciphertext space $\mathbf{C_\lambda}$
- **Enc:** $\mathbf{K_\lambda \times M_\lambda \rightarrow C_\lambda}$ (potentially randomized)
- **Dec:** $\mathbf{K_\lambda \times C_\lambda \rightarrow M_\lambda}$

Correctness:

- $\mathbf{\log|K_\lambda|, \log|M_\lambda|, \log|C_\lambda|}$ polynomial in $\boldsymbol{\lambda}$
- For all $\boldsymbol{\lambda}$, $\mathbf{k \in K_\lambda}$, $\mathbf{m \in M_\lambda}$,

$$\mathbf{Pr[\ Pr[Dec(k,\ Enc(k,m)\ ) = m\ ] = 1}$$

Statistical Distance

Given two distributions $\mathbf{D_1}$, $\mathbf{D_2}$ over a set $\mathbf{X}$, define

$$\Delta(D_1,D_2) = \tfrac{1}{2}\Sigma_x \mid Pr[D_1=x] - Pr[D_2=x] \mid$$

Observations:

$$0 \leq \Delta(D_1,D_2) \leq 1$$

$$\Delta(D_1,D_2) = 0 \iff D_1 \overset{d}{=} D_2$$

$$\Delta(D_1,D_2) \leq \Delta(D_1,D_3) + \Delta(D_3,D_2)$$

(**Δ** is a metric)

Another View of Statistical Distance

Theorem: $\Delta(D_1,D_2) \geq \varepsilon$ iff $\exists$ (potentially randomized) A s.t.

$$| \Pr[A(D_1) = 1] - \Pr[A(D_2) = 1] | \geq \varepsilon$$

Terminology: for any A,

$$| \Pr[A(D_1) = 1] - \Pr[A(D_2) = 1] |$$

is called the "advantage" of A in distinguishing D_1 and D_2

Another View of Statistical Distance

Theorem: $\Delta(D_1,D_2) \geq \varepsilon$ iff $\exists$ (potentially randomized) A s.t.

$$\left| \Pr[A(D_1) = 1] - \Pr[A(D_2) = 1] \right| \geq \varepsilon$$

To lower bound Δ, just need to show adversary A with that advantage

Examples

D_1 = Uniform distribution over $\{0,1\}^n$

- $\Pr[D_1=x] = 2^{-n}$

D_2 = Uniform subject to even parity

- $\Pr[D_2=x] = 2^{-(n-1)}$ if x has even parity, 0 otherwise

$$\Delta(D_1,D_2) = \tfrac{1}{2}\Sigma_{\text{even } x} \; |2^{-n} - 2^{-(n-1)}| + \tfrac{1}{2}\Sigma_{\text{odd } x} \; |2^{-n} - 0|$$

$$= \tfrac{1}{2}\Sigma_{\text{even } x} \, 2^{-n} + \tfrac{1}{2}\Sigma_{\text{odd } x} \, 2^{-n}$$

$$= \tfrac{1}{2}$$

Examples

D_1 = Uniform over $\{1,\dots,n\}$
D_2 = Uniform over $\{1,\dots,n+1\}$

$$\Delta(D_1,D_2) = \tfrac{1}{2}\sum_{x=1}^{n} |1/n - 1/(n+1)| + \tfrac{1}{2}|0 - 1/(n+1)|$$

$$= \tfrac{1}{2}\sum_{x=1}^{n} 1/n(n+1) + \tfrac{1}{2}\, 1/(n+1)$$

$$= \tfrac{1}{2}\, 1/(n+1) + \tfrac{1}{2}\, 1/(n+1) = 1/(n+1)$$

Statistical Security (Concrete)

Definition: A scheme **(Enc,Dec)** has **ε-statistical secrecy for d messages** if $\forall$ two sequences of messages $(m_0^{(i)})_{i\in[d]}$, $(m_1^{(i)})_{i\in[d]} \in M^d$

$$\Delta\left[\left(Enc(K,\ m_0^{(i)})\right)_{i\in[d]},\ \left(Enc(K,\ m_1^{(i)})\right)_{i\in[d]} \right] < \varepsilon$$

We will call such a scheme **(d,ε)** statistically secure

Statistical Security (Asymptotic)

Definition: A scheme **(Enc,Dec)** has **statistical secrecy for d messages** if $\exists$ negligible **ε** such that $\forall$ two sequences $(m_0^{(i)})_{i\in[d]}$, $(m_1^{(i)})_{i\in[d]}$ $\in$ M_λ^d,

$$\Delta\Big[\big(Enc(K_\lambda,\ m_0^{(i)}\ )\big)_{i\in[d]},\quad \big(Enc(K_\lambda,\ m_1^{(i)}\ )\big)_{i\in[d]}\ \Big] < \varepsilon(\lambda)$$

We will call such a scheme **d**-time statistically secure

Stateless Encryption with Multiple Messages

Ex:

$M = C = \mathbb{Z}_p$ (**p** a prime of size **2^λ**, **$\lambda=128$**)
$K = \mathbb{Z}_p^* \times \mathbb{Z}_p$
$\text{Enc}(\ (a,b),\ m) = (am + b) \bmod p$
$\text{Dec}(\ (a,b),\ c) = (c-b)/a \bmod p$

Q: Is this statistically secure for two messages?

Example

Ex:

$M = \mathbb{Z}_p$ (p a prime of size 2^λ, $\lambda=128$)
$C = \mathbb{Z}_p^2$
$K = \mathbb{Z}_p^2$
$\mathrm{Enc}((a,b), m) = (r, (ar+b) + m)$
$\mathrm{Dec}((a,b), (r,c)) = c - (ar+b)$

Random in $\mathbb{Z}_p$

Q: Is this statistically secure for two messages?

Proof of Example

Let $\mathbf{D_b}$ be distribution of $\mathbf{(\ Enc(k,m_b^{(i)})\ )_{i\in\{1,2\}}}$
Let $\mathbf{D_b'}$ be $\mathbf{D_b}$, but conditioned on $\mathbf{r_0 \neq r_1}$

Fix $\mathbf{r_0 \neq r_1,\ m_0, m_1, c_0, c_1}$

$$\Pr_{(a,b)}[ar_0+b+m_0=c_0,\ ar_1+b+m_1=c_1] = 1/p^2$$

So $\mathbf{D_0' \stackrel{d}{=} D_1'}$ $\mathbf{(\ \Delta(D_0', D_1') = 0\ )}$

Proof of Example

Lemma: $\Delta(D_1,D_2) \leq \frac{1}{2}\Pr[\text{bad}|D_1] + \frac{1}{2}\Pr[\text{bad}|D_2] + \Delta(D_1', D_2')$

Where:

- "**bad**" is some event
- $\Pr[\text{bad}|D_b]$ is probability "**bad**" when sampling from D_b
- D_b' is D_b, but conditioned on **not** "**bad**"

Proof of Lemma

$$\Delta(D_1,D_2) = \tfrac{1}{2}\Sigma_x|\ Pr[D_1{=}x] - Pr[D_2{=}x]\ |$$

$$= \tfrac{1}{2}\Sigma_{x:bad}|\ Pr[D_1{=}x\ ] - Pr[D_2{=}x\ ]\ |$$
$$+\ \tfrac{1}{2}\Sigma_{x:good}|\ Pr[D_1{=}x\ ] - Pr[D_2{=}x\ ]\ |$$

$$\leq \tfrac{1}{2}\Sigma_{x:bad}|\ Pr[D_1{=}x\ ]\ | + \tfrac{1}{2}\Sigma_{x:bad}|\ Pr[D_2{=}x\ ]\ |$$
$$+\ \tfrac{1}{2}\Sigma_{x:good}|\ Pr[D_1{=}x] - Pr[D_2{=}x]\ |$$

$$\leq \tfrac{1}{2}\ Pr[bad|D_1] + \tfrac{1}{2}\ Pr[bad|D_2] + \Delta(D_1',D_2')$$

Proof of Example

Let $\mathbf{D_b}$ be distribution of $\mathbf{(\ Enc(k,m_b^{(i)})\ )_{i\in\{1,2\}}}$
Let **bad** be when $\mathbf{r_0=r_1}$
Let $\mathbf{D_b'}$ be $\mathbf{D_b}$, but conditioned on **not bad**

$\mathbf{Pr[bad|D_b] = 1/p}$
$\mathbf{\Delta(D_0', D_1') = 0}$

Therefore, $\mathbf{\Delta(D_0, D_1) \leq 1/p \approx 2^{-\lambda}}$

Summary so Far

Stateless encryption for multiple messages ✓

But, key length grows with number of messages ✘

And, key length grows with length of message ✘

Limits of Statistical Security

Theorem: Suppose **(Enc,Dec)** has plaintext space $\mathbf{M} = \{0,1\}^n$ and key space $\mathbf{K} = \{0,1\}^t$. Moreover, assume it is $(\mathbf{d}, \mathbf{0.4999})$-secure. Then:

$$\mathbf{t \geq d\ n}$$

In other words, the key must be at least as long as the total length of all messages encrypted

Proof Idea: Compression

Use an encryption protocol to build a compression protocol

m

m′

m′ ← Comp(m)

m ← Decomp(m′)

Goal: $|m'| < |m|$

For Now: Easier Goal

$s \leftarrow \mathsf{Setup}()$

m

s

m'

$m' \leftarrow \mathsf{Comp}(s,m)$

$m \leftarrow \mathsf{Decomp}(s,m')$

Goal: $|m'| < |m|$

The Protocol

Let $\mathbf{m_0}$ be some message in $\mathbf{M}$

Setup():

- Choose random $\mathbf{k_0 \leftarrow K}$
- **Let** $\mathbf{c_1 \leftarrow Enc(k_0,m_0)}$, …, $\mathbf{c_d \leftarrow Enc(k_0,m_0)}$
- **Output** $\mathbf{(c_1,\dots,c_d)}$

In $\mathbf{M^d}$

Comp($\mathbf{(c_1,\dots,c_d), (m_1,\dots,m_d)}$ **):**

- Find $\mathbf{k,r_1,\dots,r_d}$ such that $\mathbf{c_i=Enc(k,m_i;\ r_i)\ \forall i}$
- If no such values exist, abort
- Output $\mathbf{k}$

The Protocol

Let $\mathbf{m_0}$ be some message in $\mathbf{M}$

In $\mathbf{M^d}$

Comp($(c_1,\ldots,c_d)$, $(m_1,\ldots,m_d)$):

- Find $k, r_1,\ldots,r_d$ such that $c_i = \mathrm{Enc}(k, m_i;\ r_i)\ \ \forall i$
- If no such values exist, abort
- Output k

Decomp($(c_1,\ldots,c_d)$, k):

- Compute $m_i = \mathrm{Dec}(k, c_i)$
- Output $(m_1,\ldots,m_d)$

Analysis of Protocol

If **Comp** succeeds, **Decomp** must succeed by correctness

- Since c_i=**Enc(k,**m_i**;** r_i**)**, **Dec(k,**c_i**)** must give m_i

Therefore, must figure out when **Comp** succeeds

Claim: For any sequence of messages $m_1,\ldots,m_d$, **Comp** succeeds with probability at least $1-\varepsilon$

(Probability over the randomness used by **Setup()**)

Claim: For any sequence of messages $\mathbf{m_1,\ldots,m_d}$, **Comp** succeeds with probability at least $\mathbf{1-\varepsilon}$

Proof:

- Suppose **Comp** succeeds with probability $\mathbf{1-p}$ for messages $\mathbf{m_1,\ldots,m_d}$
- Let $\mathbf{A(c_1,\ldots,c_d)}$ be the algorithm that runs $\mathbf{Comp((c_1,\ldots,c_d),\ (m_1,\ldots,m_d))}$ and outputs **1** if **Comp** succeeds

- If $\mathbf{c_i = Enc(k_0,m_i)}$, then $\mathbf{Pr[A(c_1,\ldots,c_d)=1] = 1}$
- If $\mathbf{c_i = Enc(k_0,m_0)}$, then $\mathbf{Pr[A(c_1,\ldots,c_d)=1] = 1-p}$

- By $\mathbf{(d,\varepsilon)}$-statistical security of **Enc**, $\mathbf{p}$ must be $\mathbf{\leq\varepsilon}$

Claim: For any sequence of messages $\mathbf{m_1,\dots,m_d}$, **Comp** succeeds with probability at least $\mathbf{1-\varepsilon}$

Claim: For **a random** sequence of messages $\mathbf{m_1,\dots,m_d}$, **Comp** succeeds with prob at least $\mathbf{1-\varepsilon}$

(Probability over the randomness used by **Setup()** and the random choices of $\mathbf{m_1,\dots,m_d}$)

Next step: Removing Setup

We know:

$$\Pr\left[\text{Comp succeeds}: \begin{array}{l}(c_1,\dots,c_d)\leftarrow \text{Setup}(),\\ m_i \leftarrow M\end{array}\right] \geq 1-\varepsilon$$

Therefore, there must exist *some* $(c_1^*,\dots,c_d^*)$ such that

$$\Pr[\text{Comp succeeds}: m_i \leftarrow M] \geq 1-\varepsilon$$

Define: $M' = \{(m_1,\dots,m_d)$: Comp succeeds$\}$

- Note that $|M'| \geq (1-\varepsilon)\ |M|^d$

The Protocol

$\mathbf{m \in M'}$

$\xrightarrow{\mathbf{k}}$

Find $\mathbf{k, r_1, \ldots, r_d}$ such that
$\mathbf{c_i^* = Enc(k, m_i, r_i)}$ $\forall \mathbf{i}$

For each $\mathbf{i}$,
Let $\mathbf{m_i \leftarrow Dec(k, c_i^*)}$
Output $\mathbf{(m_1, \ldots, m_d)}$

By previous analysis,
- Alice always successfully compresses
- Bob always successfully decompresses

Final Touches

Can compress messages in $\mathbf{M'}$ into keys in $\mathbf{K}$

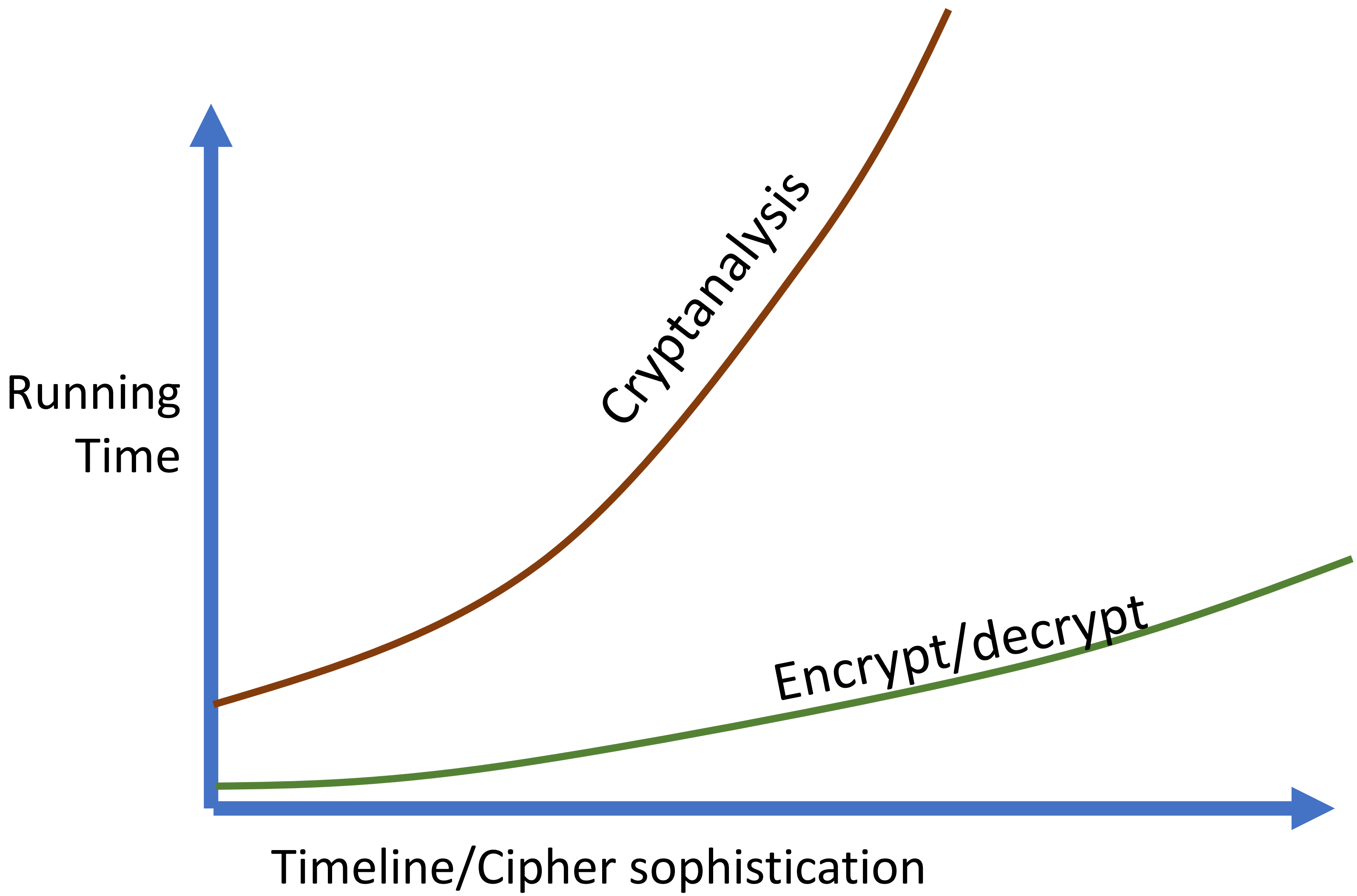
Therefore, it must be that $|M'| \leq |K|$

$$
\begin{aligned}
\textbf{Meaning } t &= \log |K| \\
&\geq \log |M'| \\
&\geq \log [\,(1-\varepsilon)\,|M|^d\,] \\
&= d \log |M| + \log [1-\varepsilon] \\
&= dn + \log [1-\varepsilon] \\
&\geq dn \ \text{(as long as } \varepsilon<1/2\text{)}
\end{aligned}
$$

Takeaway

If you don't want to physically exchange keys frequently, you cannot obtain statistical security

So, now what?


Cryptanalysis
Running
Time
Encrypt/decrypt
Timeline/Cipher sophistication

Computational Security

We are ok if adversary takes a really long time

Only considered attack for adversaries that don’t take too long

How Long Is Ok?

Practice:

- Lifetime of data/person/civilization/universe

- Typically something like 2^{80}, 2^{128}, or maybe 2^{258}
  - Lifetime of universe in nanoseconds: 2^{58}
  - Number of atoms in known universe: 2^{265}

How Long Is Ok?

Theory:

- Maybe things will change as technology improves
- Want a more conceptual answer
- Absolute constants unsatisfactory
- Instead, consider an attack if time bounded by polynomial function

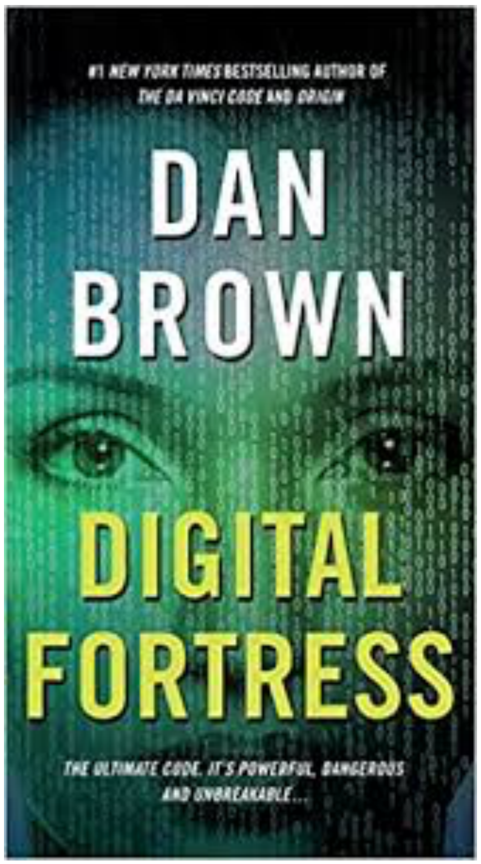
Brute Force Attacks

Simply try every key until find right one

If keys have length $\boldsymbol{\lambda}$, $\mathbf{2}^{\lambda}$ is upper bound on attack

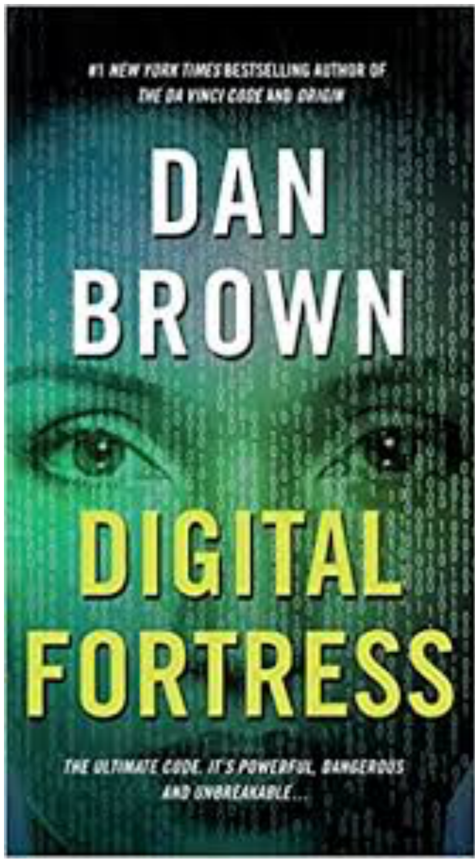
Not always applicable. When?

Holiwudd Criptoe!



[TRANSLTR]'s three million processors would all work in parallel ... trying every new permutation as they went

Holiwudd Criptoe!



"What's the longest you've ever seen TRANSLTR take to break a code?"

"About an hour, but it had a ridiculously long key—ten thousand bits"

Reminders

HW1 Due Feb 20th

Project 1 to be released hopefully this afternoon