

COS433/Math 473: Cryptography

Mark Zhandry

Princeton University

Spring 2020

What is Cryptography?

What is Cryptography

Concise Oxford English Dictionary: *“the art of writing or solving codes”*

Merriam-Webster: *“the enciphering and deciphering of messages in secret code or cipher”*

Wikipedia: *“the practice and study of techniques for secure communication in the presence of third parties called adversaries”*

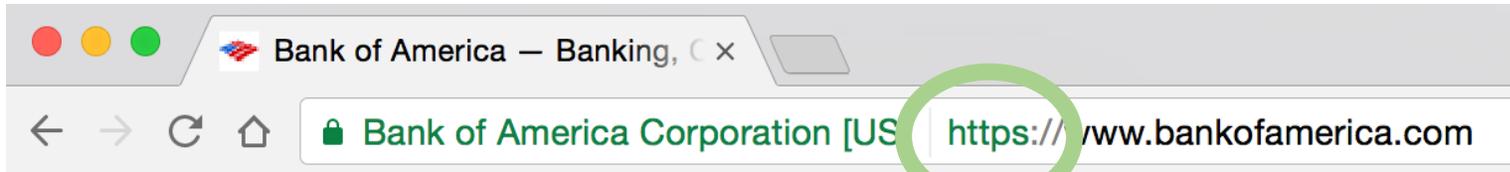
None of these capture the true breadth of the field

My Definition

Cryptography is about using secrets
to solve interesting tasks

(still doesn't capture everything)

Cryptography Is Everywhere



Sign in to add another account

A sign-in form for a user named Mark Zhandry. It features a back arrow, a profile picture with the letter 'M', the name 'Mark Zhandry', and the email 'mzhandry@gmail.com'. There is a text input field for the password, a blue 'Sign in' button, and a link for 'Forgot password?'.

A Long & Rich History

Dates back almost 4000 years

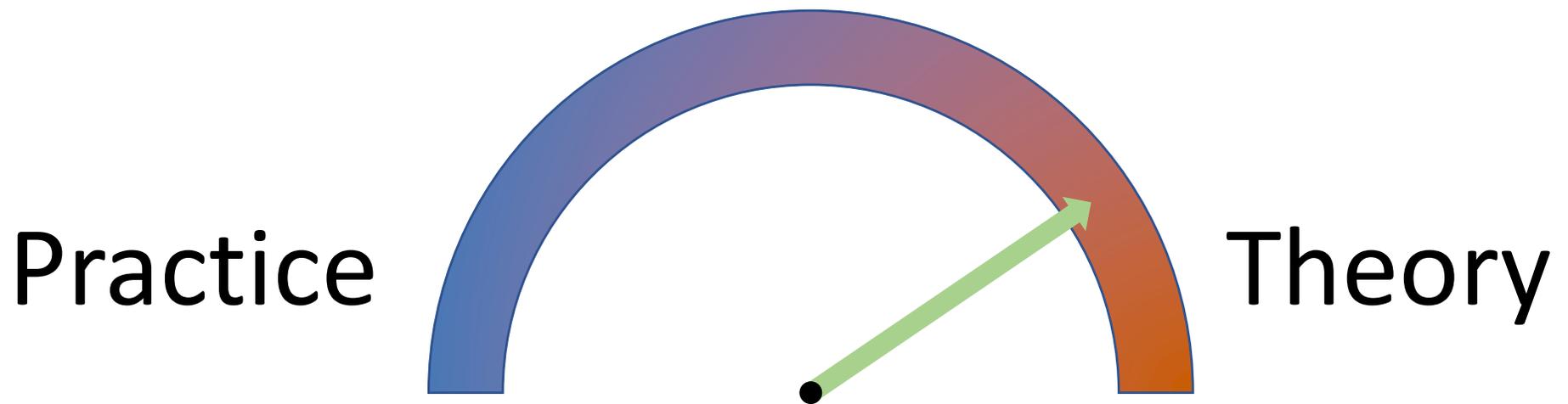
Important historical consequences

- 1587 – Babington Plot
- WWI – Zimmermann Telegram
- WWII – Enigma

Intimately tied to development of modern computer

- First program written for Atlas supercomputer
- First magnetic core memories, high-speed tape drives, all-transistor computers, desktop-sized computers, remote workstations all built based on NSA orders

COS 433



Inherent to the study of crypto

- Working knowledge of fundamentals is crucial
- Cannot discern security by experimentation
- Proofs, reductions, probability are necessary

COS 433

What you should expect to learn:

- Foundations and principles of modern cryptography
- Core building blocks
- Applications

Bonus:

- Debunking some Hollywood crypto
email me scenes from movies/shows!
- Better understanding of crypto news

COS 433

What you will **not** learn:

- Hacking
- Implementing crypto
- How to design secure **systems**
- Viruses, worms, buffer overflows, etc

Administrivia

Course Information

Instructor: Mark Zhandry

TAs: Ben Kuykendall
Jiaxin (Ernest) Guan

Lectures: TuTh 11:00-12:20pm, Friend 008

Webpage: cs.princeton.edu/~mzhandry/2020-Spring-COS433/

Office Hours: please fill out HW0 poll

Piazza

<https://piazza.com/class/k62c87yjhgl51n>

Main channel of communication

- Course announcements
- Discuss homework problems with other students
- Find project/study groups
- Ask content questions to instructors, other students

Prerequisites

- Ability to read and write mathematical proofs
- Familiarity with algorithms, analyzing running time, proving correctness, O notation
- Basic probability (random variables, expectation)

Helpful:

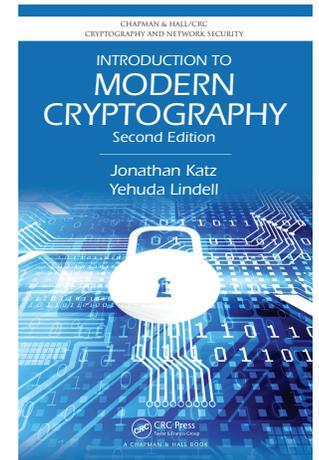
- Familiarity with NP-Completeness, reductions
- Basic number theory (modular arithmetic, etc)

Reading

No required text

But highly recommend:

Introduction to Modern Cryptography
by Katz, Lindell



For each lecture, page numbers for 2nd edition will be posted on course website

Grading

40% Homeworks

- ~1 per week
- **4 late days**
- Only typed solutions, submission instructions TBA
- Collaboration encouraged, but write up own solutions

30% Projects

- More details next week

30% Take-home Final

- Individual

Classroom Policies

Please stop me if you have any questions

Please come to class to be engaged and to learn

- Notes for each lecture will be added to the webpage
- I don't take attendance
- Don't be on Facebook, working on assignments, etc

Feel free to call me "Mark", "Professor", "Hey You", etc, though "Mark" is preferred

Approximate Course Outline

Week 1: Pre-modern crypto (\leq ~1950s)

Weeks 2-6: Foundations of modern cryptography

- Crypto theory
- Symmetric key cryptography

Weeks 7-12: Public key cryptography

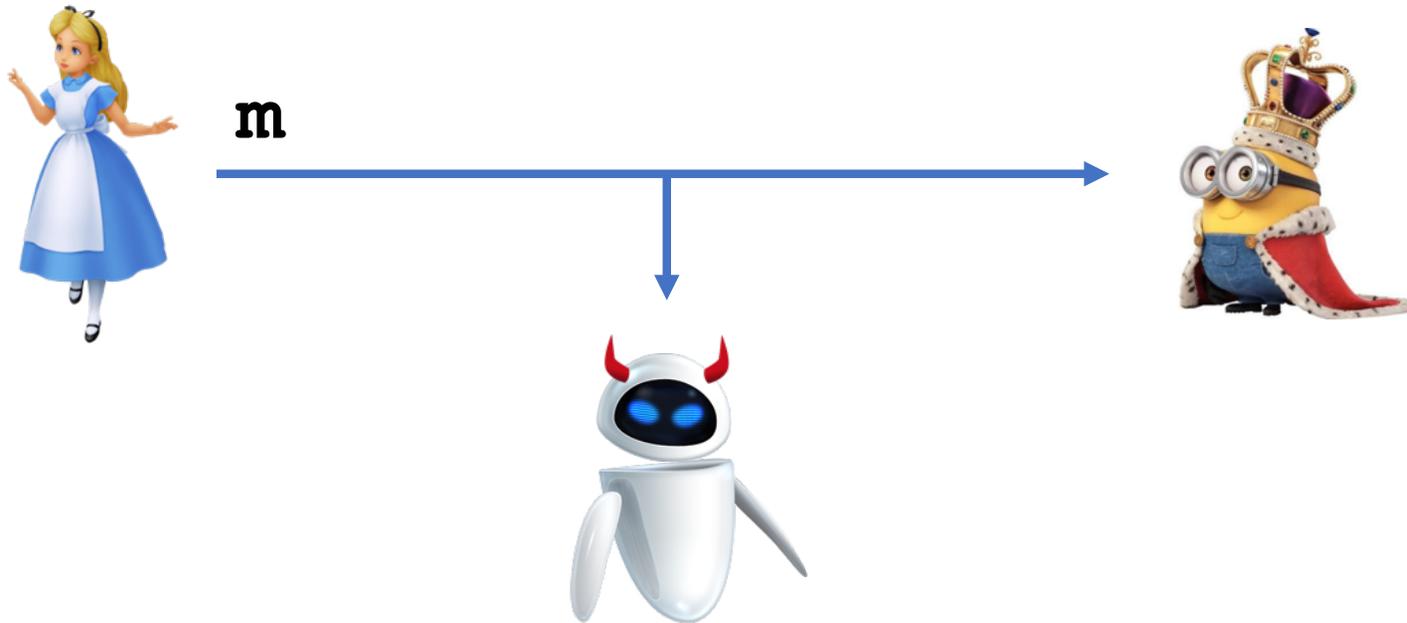
Today

“Pre-modern” Crypto Part I:
Pencil & Paper Ciphers

Pre-modern Cryptography

1900 B.C. – mid 1900's A.D.

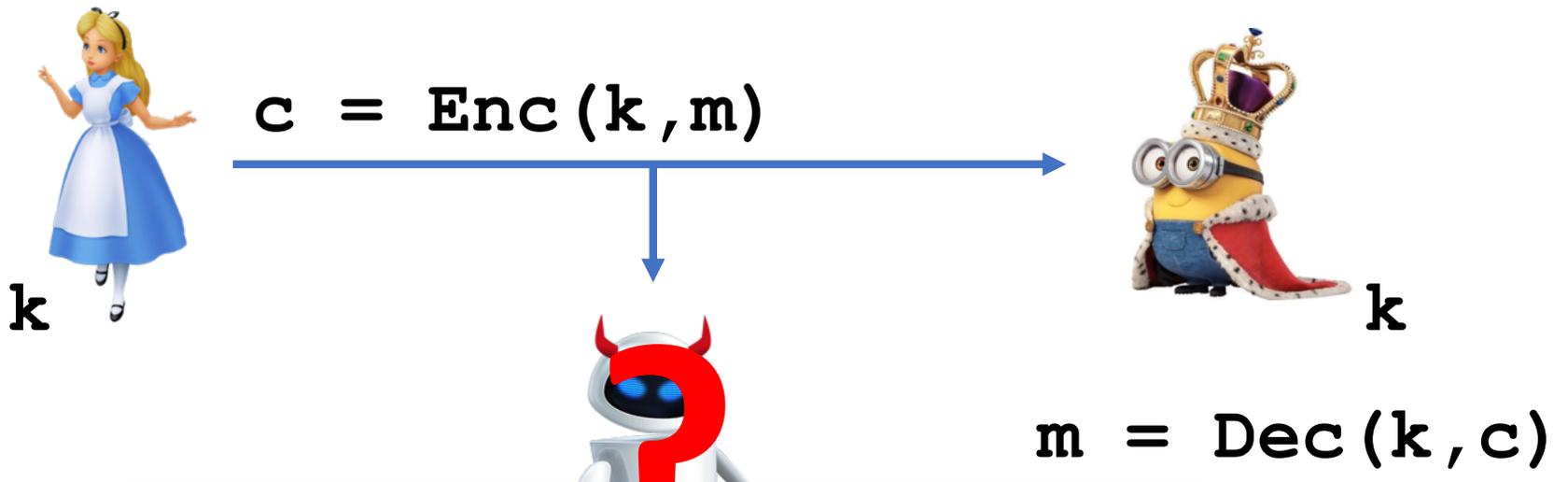
With few exceptions, synonymous with **encryption**



Pre-modern Cryptography

1900 B.C. – mid 1900's A.D

With few exceptions, synonymous with **encryption**



For our discussions, assume **Enc**,
Dec known, only **k** is secret

Ancient Crypto

1900 BC, Egypt



1500 BC, Mesopotamia



50 B.C. – Caesar Cipher

Used by Julius Caesar

Alphabet shift by 3

a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C

Example:

plaintext: **super secret message**

ciphertext: **VXSHU VHFUHW PHVVDJH**

Caesar not a true cipher: what's the secret key?

Generalization: Shift Ciphers

Shift by fixed, secret increment ($k = 0, \dots, 25$)

Some examples:

- Shift by 1: Augustus Caesar; Jewish mezuzah
- Shift by 3: Caesar Cipher
- Shift by 13: ROT13

Sometimes also called “Caesar ciphers”

Security of Shift Ciphers?

Problem: only 26 possibilities for key

“Brute force” attack:

- Try all 26 possible shifts
- For each shift, see if something sensible comes out

Example Brute Force Attack

Ciphertext: **HJETG HTRGTI BTHHPVT**

Key	Plaintext
0	HJETG HTRGTI BTHHPVT
1	IKFUH IUSHUJ CUIIQWU
2	JLGI VI JVTIVK DVJJRXV
3	KMHWJ KWUJWL EWKKS YW
4	LNIXK LXVKXM FXLLTZX
5	MOJYL MYWLYN GYMMUAY
6	NPKZM NZXMZO HZNNVBZ
7	OQLAN OAYNAP IAOWCA
8	PRMBO PBZOBQ JBPPXDB
9	QSNCP QCAPCR KCQQYEC
10	RTODQ RDBODS LDRRZFD
11	SUPER SECRET MESSAGE
12	TVQFS TFDSFU NFTTBHF

Key	Plaintext
13	UWRGT UGETGV OGUUCIG
14	VXSHU VHFUHW PHVVDJH
15	WYTIV WIGVIX QIWWEKI
16	XZUJW XJHWJY RJXXFLJ
17	YAVKX YKIXKZ SKYYGMK
18	ZBWLY ZLJYLA TLZZHNL
10	ACXMZ AMKZMB UMAAIOM
20	BDYNA BNLANC VNBBJPN
21	CEZOB COMBOD WOCKQO
22	DFAPC DPNCPE XPDDL RP
23	EGBQD EQODQF YQEEMSQ
24	FHCRE FRPERG ZRFFNTR
25	GIDSF GSQFSH ASGGOUS

Security of Shift Ciphers?

Problem: only 26 possibilities for key

“Brute force” attack:

- Try all 26 possible shifts
- For each shift, see if something sensible comes out

To avoid brute force attacks, need large key space

- On modern hardware, typically need $\#(\text{keys}) \geq 2^{80}$
(Usually choose at least 2^{128} , 2^{256})

Generalization: Substitution Ciphers

Apply fixed permutation to plaintext letters

a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
F	M	S	G	Y	U	J	B	T	P	Z	K	E	W	L	Q	H	V	A	X	R	D	N	C	I	O

Example:

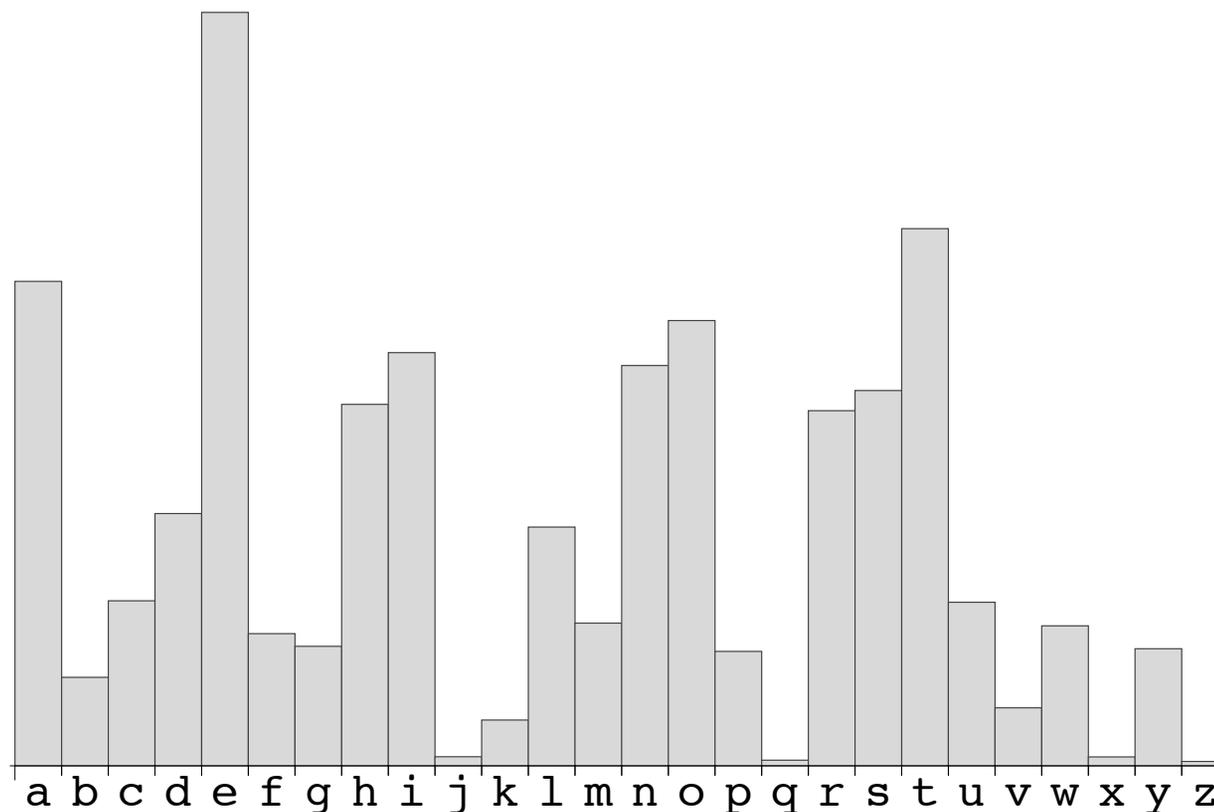
plaintext: **super secret message**
ciphertext: **ARQYV AYSVYX EYAAFJY**

Number of possible keys?

$26! \approx 2^{88}$  brute force attack expensive

800's A.D. – First Cryptanalysis

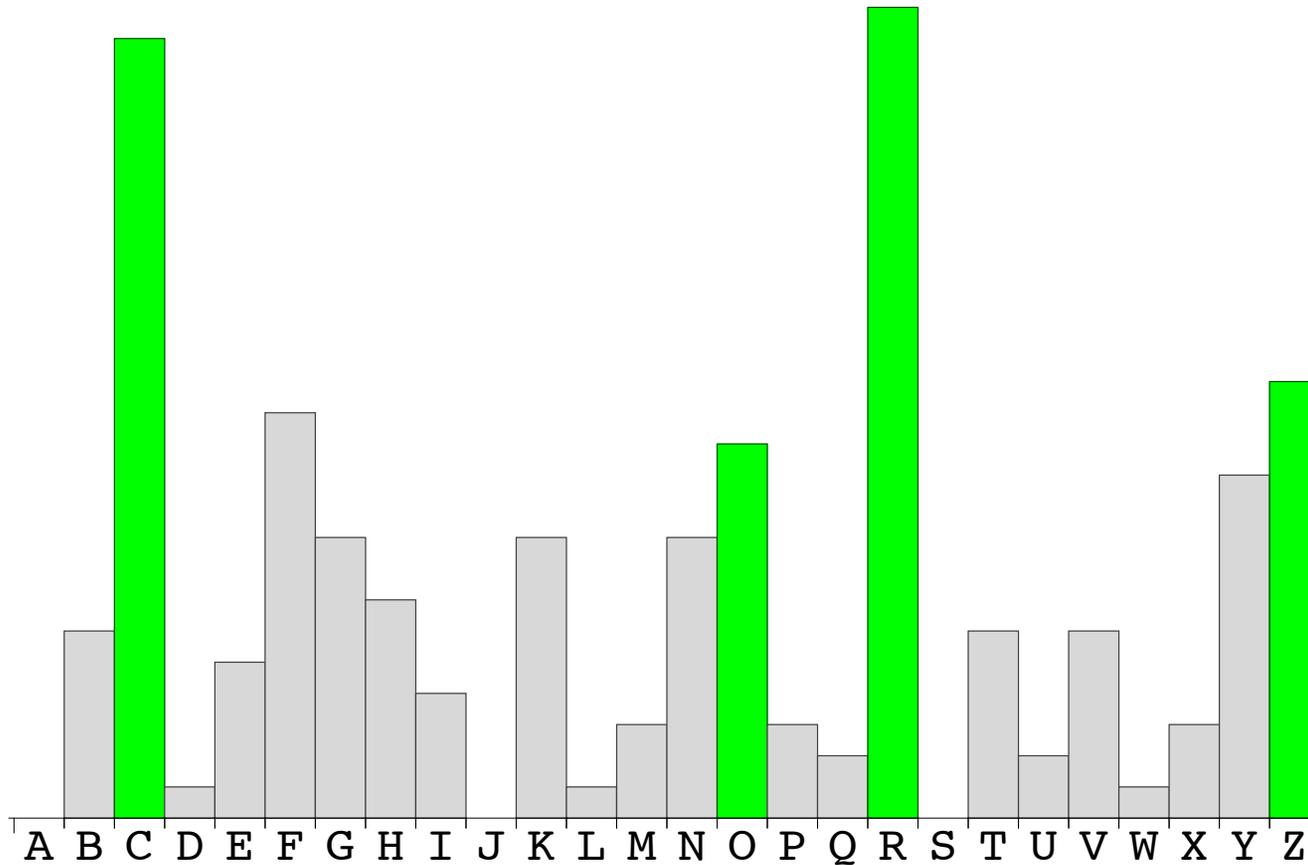
Al-Kindi – Frequency Analysis: some characters are more common than others



Example

BOFC HNR Z NHMNCYCHCYOF KYIVRG CO RFKOB
NRFNICYPR BZCZ, RPRF CVOHXV CVRE ZGR
GRNYTYRFC CO Z MGHCR WOGKR ZCCZKU.
YFBRRB, ME KOHFCYFX TRCCRGN ZFB KODIZGYFX
CO CEIYKZT CRQC, EOH KZF GRKOPRG CVR
ITZYFCRQC ZN LRTT ZN CVR URE

Example



Reasonable conjecture:
 $e \rightarrow R, t \rightarrow C, a \rightarrow Z, o \rightarrow O$

Example

BoFt HNe a NHMNTYtHTYoF KYIVeG to eFKoBe
NeFNyTYPe **Bata** ePeF tVoHXV tVeE aGe
GeNYTYeFt to a MGhte WoGKe **attaku**.
YFBeeB, ME KOHFtYFX TetteGN aFB KoDIAGYFX
to tEIYKaT teQt, EoH KaF GeKoPeG **tve**
ITaYFteQt an LetT an tve UeE

Maybe "data"?

Maybe "attack"?

Probably "the"

a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
Z				R										O					C						

Example

doFt HNe a NHMNTYtHTYoF cYIheG to eFcode
NeFNyTPE data, ePeF thoHXh theE aGe
GeNYTYeFt to a MGHte WoGce attack.
YFdeed, ME coHFtYFX TetteGN aFd coDIaGYFX
to tEIYcaT teQt, EoH caF GecoPeG the
ITaYFteQt aN LetT aN the keE

“as”?

“and”?

“are”?

“encode”?

a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
Z		K	B	R			V			U				O						C					

Example

“use”?

dont **Hse** a **SHMstYtHTY**on **cYI**her to encode
sens**YtPE** data, **ePen** tho**HXh** the**E** are
res**YTY**ent to a **MrHte** **Worce** attack.

Yndeed, **ME** co**HntYnX** **TetteGs** and co**DIarYnX**
to **tEYcaT** te**Qt**, **EoH** can **recoPer** the
ITaYnteQt as **LeTT** as the ke**E**

“indeed”?

“even”?

“force”?

“recover”?

a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
Z		K	B	R			V			U			F	O				G	N	C					

Example

don't use a substitution cipher to encode sensitive data, even though they are resistant to a brute force attack.

indeed, if you count the letters and compare them to the ciphertext, you can recover the plaintext as long as the key

a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
Z		K	B	R	W		V	Y		U			F	O			G	N	C	H	P				

Example

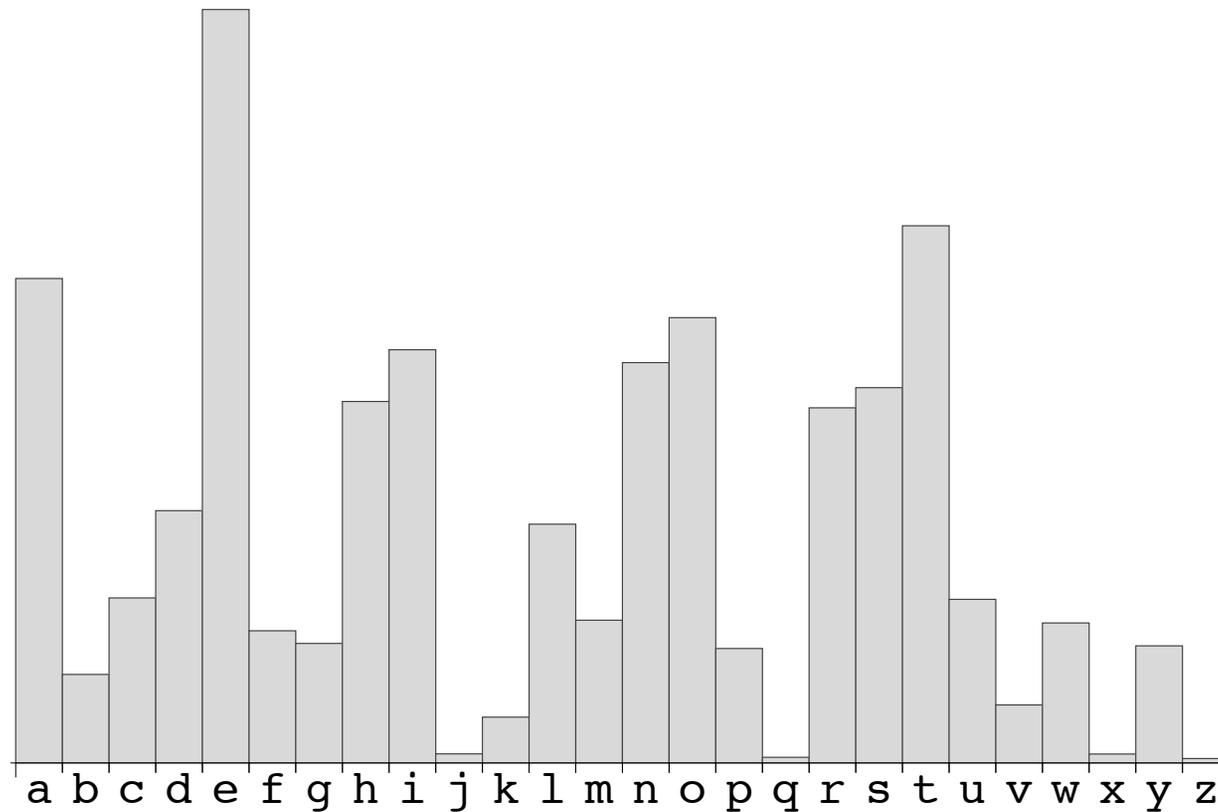
don't use a substitution cipher to encode sensitive data, even though they are resilient to a brute force attack.

indeed, by counting letters and comparing to typical text, you can recover the plaintext as well as the key

a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
Z	M	K	B	R	W	X	V	Y		U	T	D	F	O	I		G	N	C	H	P	L	Q	E	

Problem with Substitution

Differing letter frequencies reveal a lot



Substitution Cipher Variants

Polybius Square

	1	2	3	4	5
1	a	b	c	d	e
2	f	g	h	ij	k
3	l	m	n	o	p
4	q	r	s	t	u
5	v	w	x	y	z

plaintext: s u p e r s e c r e t m e s s a g e

ciphertext: 4345351542 431513421544 32154343112215

Problem?

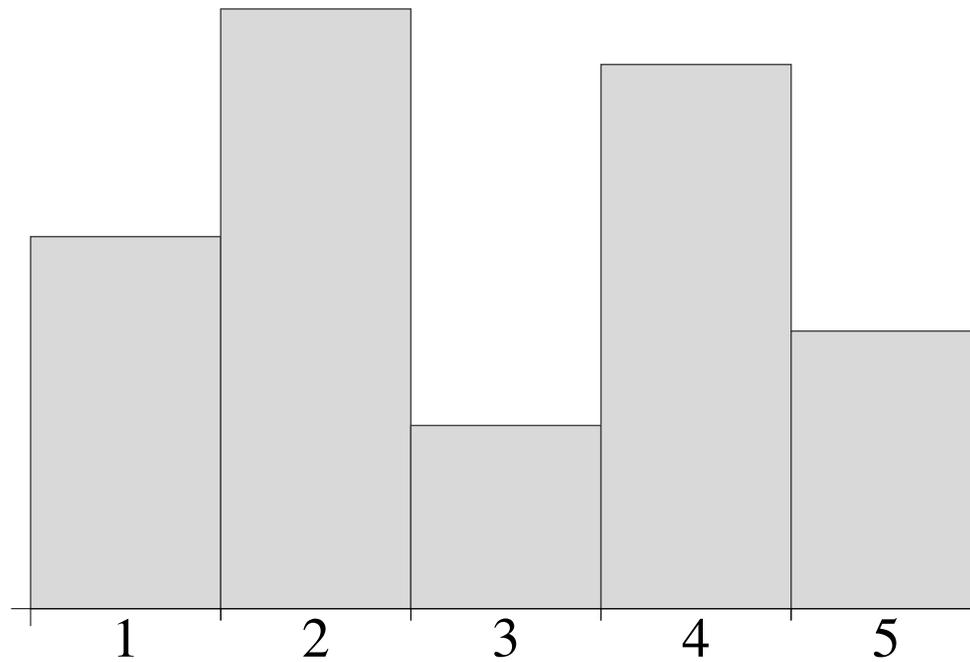
Keyed Polybius Square

	1	2	3	4	5
1	y	n	r	b	f
2	d	l	w	o	g
3	s	p	a	t	k
4	h	v	ij	x	c
5	q	u	z	e	m

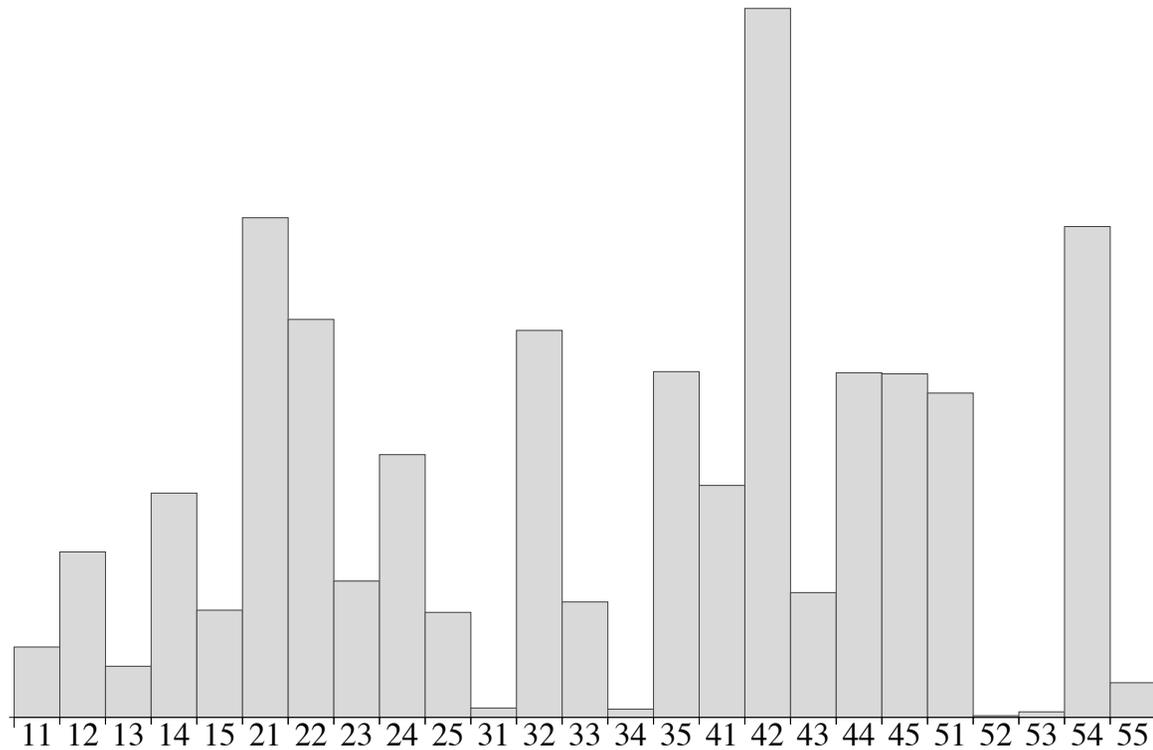
plaintext: s u p e r s e c r e t m e s s a g e

ciphertext: 3152325413 315445135434 55543131332554

Frequency of Polybius?



Frequency of Polybius?



General Alphabets

Ptxt and ctxt symbols need not be the same

- ctxt symbols can be letters, (tuples of) numbers, etc.
- ptxt symbols can also numbers, bits, bytes

In general, changing ctxt alphabet doesn't affect security of cipher

- Keyed Polybius = Un-keyed Polybius + Substitution

Other reasons to change ciphertext alphabet?

Pigpen Cipher

A	B	C
D	E	F
G	H	I

J	K	L
M	N	O
P	Q	R

	S	
T	U	
	V	

	W	
X	Y	
	Z	

> X > □ □ □ □ V > □ □ V □ □ >
 X M A R K S T H E S P O T

Polygraphic Substitution

Frequency analysis requires seeing many copies of the same character/group of characters

Idea: encode **d = 2, 3, 4**, etc characters at a time

- New alphabet size: **26^d**
- Symbol frequency decreases:
 - Most common digram: “th”, 3.9%
 - trigram: “the”, 3.5%
 - quadrigram: “that”, 0.8%
- Require much larger ciphertext to perform frequency analysis

Polygraphic Substitution

Example: Playfair cipher

- Invented by Sir Charles Wheatstone in 1854
- Used by British until WWII

Y	N	R	B	F
D	L	W	O	G
S	P	A	T	K
H	V	IJ	X	C
Q	U	Z	E	M

Polygraphic Substitution

Example: Playfair cipher

- Invented by Sir Charles Wheatstone in 1854
- Used by British until WWII

Y	N	R	B	F
D	L	W	O	G
S	P	A	T	K
H	V	IJ	X	C
Q	U	Z	E	M

TH

- To encode, choose opposite corners of rectangle

Polygraphic Substitution

Example: Playfair cipher

- Invented by Sir Charles Wheatstone in 1854
- Used by British until WWII

Y	N	R	B	F
D	L	W	O	G
S	P	A	T	K
H	V	IJ	X	C
Q	U	Z	E	M

TH → **XS**

- To encode, choose opposite corners of rectangle
- Additional rules for repeats, digrams in same row, etc

Polygraphic Substitution

Limitations:

- For small **d**, frequency analysis still possible given enough ciphertext material
- For large **d**, need $> 26^d$ bits to write down general substitutions
 - Impractical to use arbitrary permutations for large **d**
 - Some tricks (like Playfair) possible to reduce key size while minimizing risk of frequency analysis

Homophonic Substitution

In principle, by using sufficiently large ciphertext alphabet, character frequencies can be made \approx uniform
 \Rightarrow Thwarts vanilla frequency analysis

However, still possible to cryptanalyze

- Frequency analysis on tuples of letters will still be non-uniform

Homophonic Substitution

Example: "Grand Chiffre" (Great Cipher)

A	B	C	D	E	F	G	H	I	J	K	L	M
811	117 258	219	407	511	555	340	141 163	205	518		820	279 448
702	359 500	338	595	733	527	618	284 164	436	639			615 827
genera.l.uax	35	lieu, x	668	Ob	19	proeque	801					
gens	35	limites	708	obei	39	proten, dre, tion	30					
ger	575	livre	728	objet, s	69	preteate	841					
ges	115	le Roy de	758	oblig, er, ation	89	pri	881					
gla	155	le Prince, de	798	observ, er, ation	129	principal, uax	32					
gle	215	le Duc de	838	obstacle, s	179	prisonnier, s	132					
gli	275	le Marquis de	878	obtenir	229	pro	162					
glo, ure	335	le Baron de	918	oc, ca, sion	249	prochain	202					
gna	375	le Sieur de	958	ocup, er	289	profit, er	262					
gne	845	loin	79	of	349	projet, s	282					
gni	485	Lon	319	office, ier, s	429	proposition, s	382					
gno	525	Lors	189	offre, s	449	provision, s	422					
gouvern, er, ment	10	luy	848	oient	499	prouv	442					
gra, ce	405	Ma	868	oir	529	pru	462					
grand	525	me	298	oirc	579	publi, er, c	512					
gre	585	mo	779	oit	629	puis, sance	572					
gri	625	mu	379	ol	669	Qu	642					
gro	665	mo	439	om	729	qua	672					
gua	695	mu	489	on, s	779	qualite	722					
gue	735	magasin, s	519	ont	739	quand	742					
guerre	825	main, s	549	op, pose, ition	819	quantite	762					
gui, de, s	895	mais	159	or	849	quarente	782					
ha	26	maitre, s	609	ordinaire, s	899	quart, ier, s	822					
be	56	mal, ade, s, je, s	639	ordonn, er	909	quatre	842					
bi	156	mand, er	679	ordre, s	969	que	862					
bo	216	maniere, s	719	or, s, t	1009	quel, le, s	882					
bu	266	manque, r	759	or, t	1059	question, s	902					
baut	326	marcbe, s	799	ou, r	1109	qui	922					
babi, t, le, tano	486	marqu, e, r	839	outr	1159	qu'il	942					
beur, e, s	546	marche, s, ux	879	ouvr	1209	quinze	962					
bi	796	mauvais	919	Pa	1259	quo, n	982					
bi	856	meilleur	959		1309		1002					

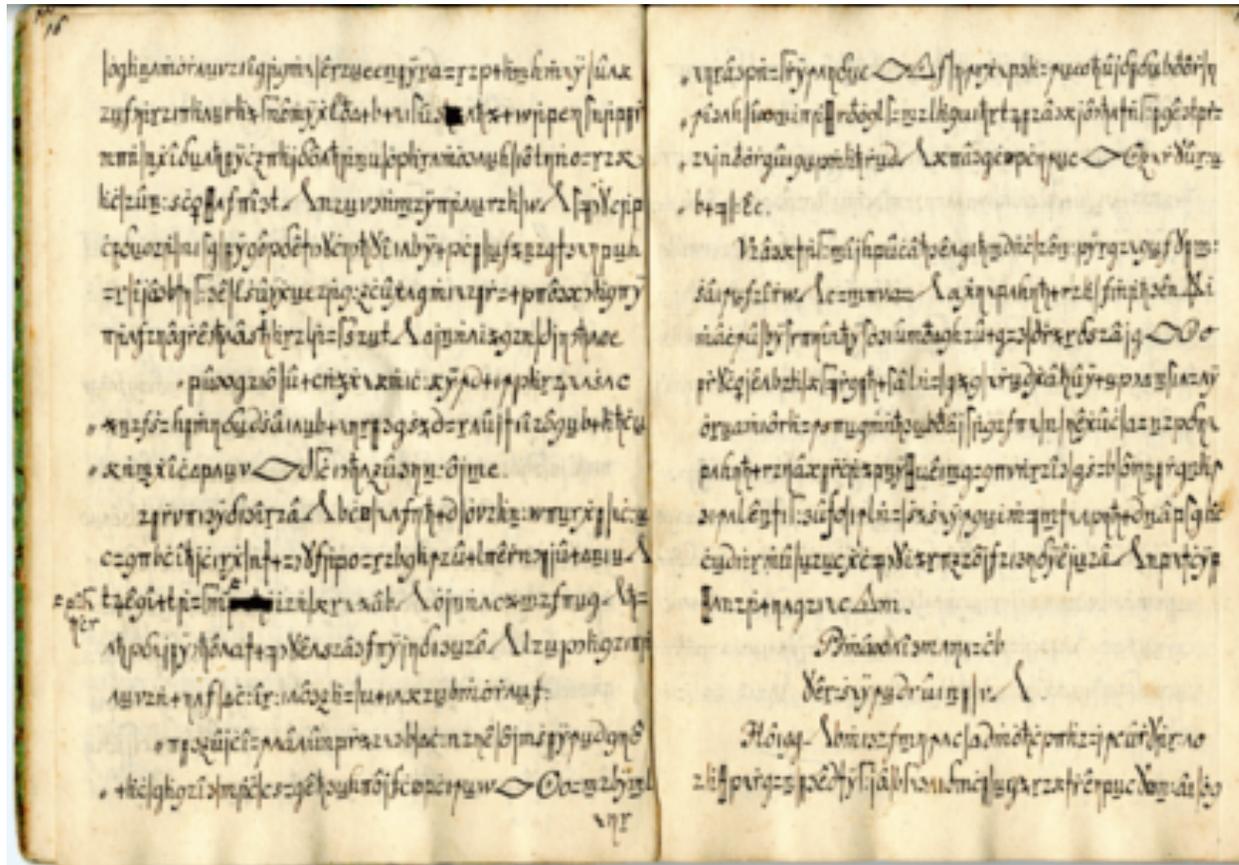
Homophonic Substitution

Example: “Grand Chiffre” (Great Cipher)

- Developed in 1600’s, used by Louis XIV
- Remained unbroken for 200 years
- Combination of polygraphic and homophonic
- 1890’s - finally cracked by Étienne Bazeries
 - Gussed that “124-22-125-46-345” stood for “les ennemies”
 - From there, things unraveled

Homophonic Substitution

Example: Copiale cipher



Homophonic Substitution

Example: Copiale cipher

- 105-page encrypted book written in 1730's
- Secret society of German ophthalmologists
 - Believed to be Freemasons whose rites had been banned by the pope
- Not broken until 2011 with help of computers

Polyalphabetic Substitution

Use a different substitution for each position

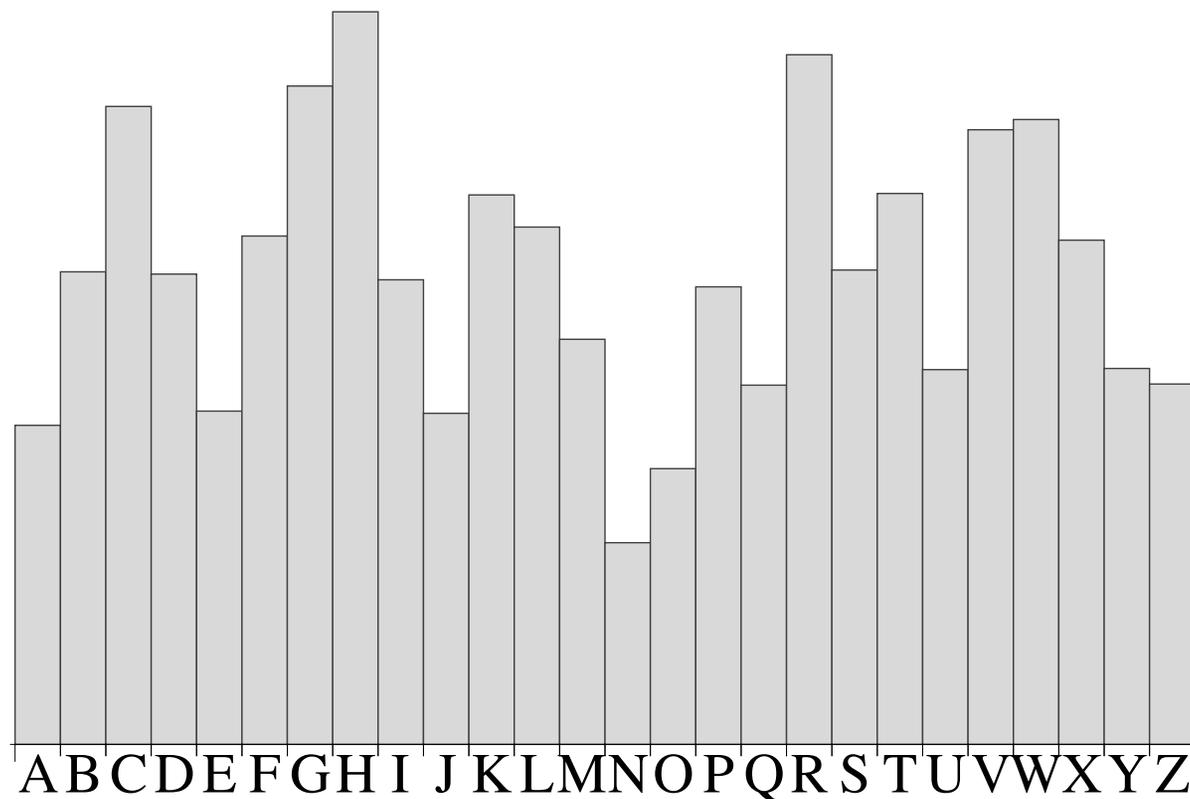
Example: Vigenère cipher

- Sequence of shift ciphers defined by keyword

keyword:	crypt	ocrypt	ocrypto
plaintext:	super	secret	message
ciphertext:	ULNTK	GGTPTM	AGJQPZS

Polyalphabetic Substitution

Vanilla frequency analysis gives average of several substitution ciphers



Cryptanalysis of Vigenère

Suppose we know keyword length

- Group letters into n buckets, each bucket encrypted using the same shift
- Perform frequency analysis on each bucket

Suppose we don't know keyword length

- Brute force: try several lengths until we get the right one
- Improvement: Kasiski examination, superposition

Superposition

Compare shifts of ciphertext, looking for shifts containing many matches

Example: shift by 1

CTYCGSTTYCVOPRQBTBATYCLOURAPGBGIAPGQCEAPGG
CTYCGSTTYCVOPRQBTBATYCLOURAPGBGIAPGQCEAPGG



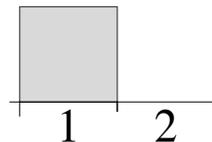
Superposition

Compare shifts of ciphertext, looking for shifts containing many matches

Example: shift by 2

CTYCGSTTYCVOPRQBTBATYCLOURAPGBGIAPGQCEAPGG

CTYCGSTTYCVOPRQBTBATYCLOURAPGBGIAPGQCEAPGG

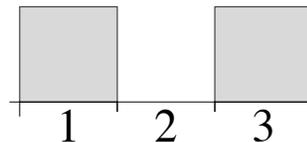


Superposition

Compare shifts of ciphertext, looking for shifts containing many matches

Example: shift by 3

CTYCGSTTYCVOPRQBTBATYCLOURAPGBGIAPGQCEAPGG
CTYCGSTTYCVOPRQBTBATYCLOURAPGBGIAPGQCEAPGG



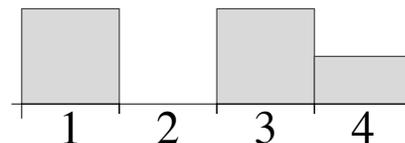
Superposition

Compare shifts of ciphertext, looking for shifts containing many matches

Example: shift by 4

CTYCGSTTYCVOPRQBTBATYCLOURAPGBGIAPGQ**CEAPGG**

CTYCGSTTYCVOPRQBTBATYCLOURAPGBB**GIAPGQCEAPGG**



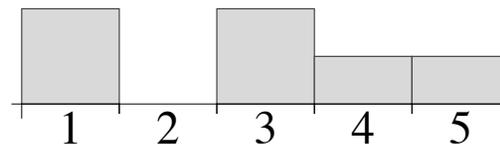
Superposition

Compare shifts of ciphertext, looking for shifts containing many matches

Example: shift by 5

CTYCGSTTYCVOPRQBTBATYCLOURAPGBGIAPGQCEAPGG

CTYCGSTTYCVOPRQBTBATYCLOURAPGBGIAPGQCEAPGG

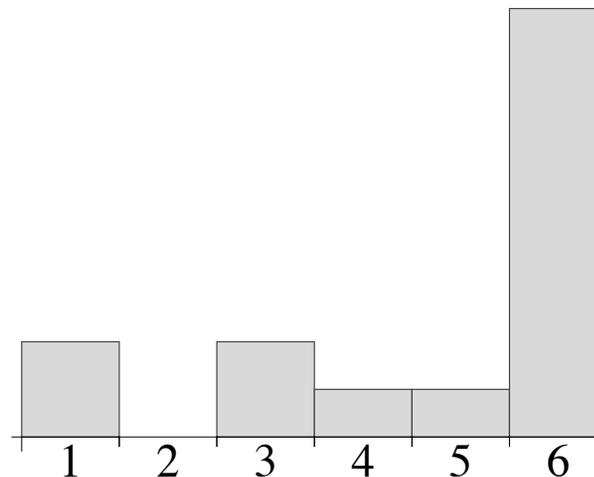


Superposition

Compare shifts of ciphertext, looking for shifts containing many matches

Example: shift by 6

CTYCGSTTYCVOPRQBTBATYCLOURAPGBGIAPGQCEAPGG
CTYCGSTTYCVOPRQBTBATYCLOURAPGBGIAPGQCEAPGG

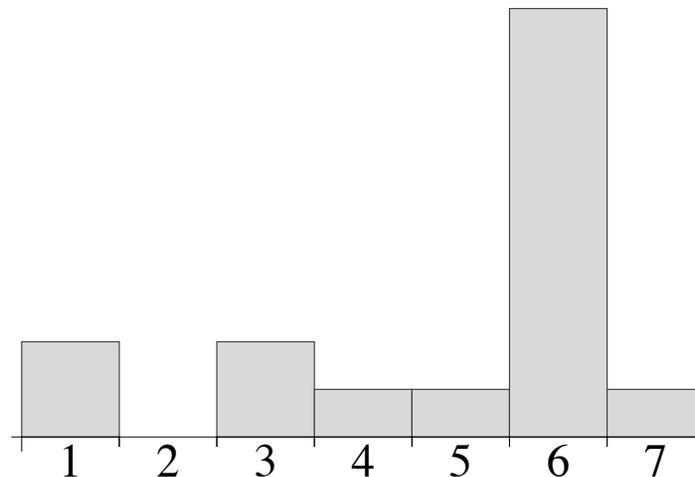


Superposition

Compare shifts of ciphertext, looking for shifts containing many matches

Example: shift by 7

CTYCGSTTYCVOPRQBTBATYCLOURAPGBGIA**PGQ**CEAPGG
CTYCGSTTYCVOPRQBTBATYCLOURAPGBGIA**PGQ**CEAPGG



Superposition

Why does it work?

For shifts that are multiples of key size:

- Both bottom and top ciphertexts encrypted with same key
- **$\#(\text{ctxt matches}) = \#(\text{ptxt matches})$**
 - $\approx |\text{ptxt}| * \text{col. prob. for English}$**
 - $\approx |\text{ptxt}| * 0.065$**

Superposition

Why does it work?

For shifts that are NOT multiples of key size:

- Both bottom and top ciphertexts encrypted with “independent” shifts
- Probability of a match at any position is **$1/26 \approx 0.038$**
- **$\#(\text{ctxt matches}) \approx |\text{ptxt}| * 0.038$**

The One-Time Pad

Vigenère on steroids

- Every character gets independent substitution
- Only use key to encrypt one message,
key length \geq message length

keyword:	agule	melpqw	gnspemr
plaintext:	super	secret	message
ciphertext:	SAIPV	EINGUP	SRKHESR

No substitution used more than once, so frequency analysis is impossible

The One-Time Pad

1882: described by Frank Miller for the telegraph

- Words and phrases first converted to 5-digit numbers using a codebook
- Key = sequence of “shift-numbers” to be added to resulting digits

1919: Patent for Vernam cipher

- Map characters to 5-bit strings using Baudot code
- Bitwise XOR with key = random bit string

Limitations of One-time Pad

Need extremely large random keys and secure way to transmit them!

5-UCO British OTP system (WWII)

- Key tape for single unit cost £5,000 a year
(~\$300k in 2020 dollars)

German GEE (WWII)

- Key's not truly random, cryptanalyzed by US Army

Russian diplomatic OTP (WWII, Cold War)

- Tapes occasionally re-used, successful cryptanalysis by US and UK intelligence

Cryptanalysis of OTP

Try to encrypt two messages, security will fail

$$\begin{aligned} \mathbf{Enc(k, m_0)} & - \mathbf{Enc(k, m_1)} \\ & = (\mathbf{k + m_0}) - (\mathbf{k + m_1}) \\ & = \mathbf{m_0 - m_1} \end{aligned}$$

Enough redundancy in English text to usually recover messages from difference

Transposition Ciphers

Transposition Ciphers

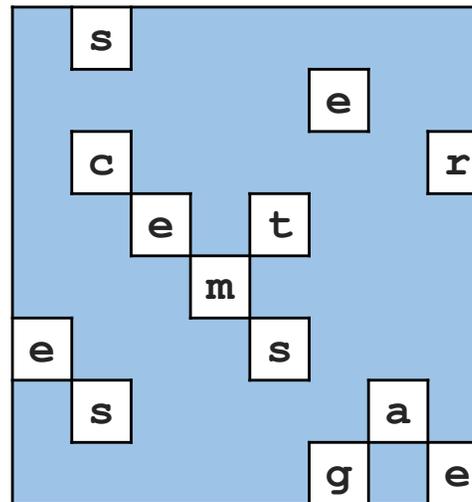
Shuffle plaintext characters

Greek Scytal (600's B.C.)



<https://commons.wikimedia.org/wiki/File:Skytale.png>

Grille (1500's A.D.)



a	s	h	o	e	v	q	k
g	i	p	c	e	e	f	j
e	c	n	i	d	z	w	r
g	i	e	b	t	e	b	o
k	c	d	m	i	z	d	p
e	b	i	d	s	h	e	r
n	s	d	u	r	e	a	v
h	k	e	g	u	g	a	e

Aside: Steganography

Hiding the fact that a message is even being sent

Many examples

- Invisible ink
- Microdots
- Blinking Morse-code
- Images in low-order color bits
- Delays in network packets
- Differing typefaces

Holiwudd Criptoe!



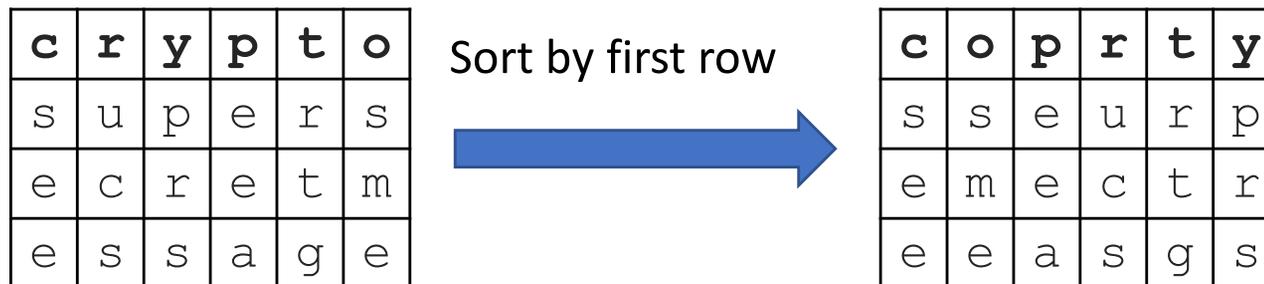
Do you know what a Vigenère cipher is? It's a form of encryption that allows a person to hide messages inside regular texts.

Column Transposition

key: **crypto**

ptxt: **supersecretmessage**

Encryption:



ctxt: **SEESMEEEAUCSRTGPRS** (read off columns)

Cryptanalysis:

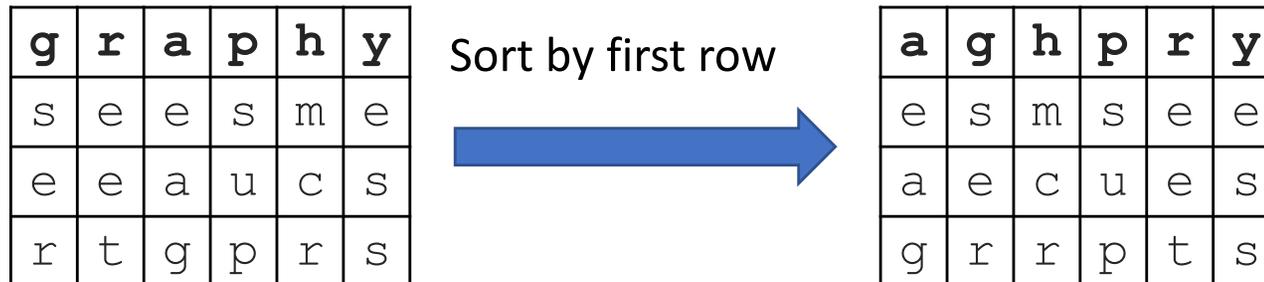
- Guess key length, reconstruct table
- Look for anagrams in the rows

Double Column Transposition

key: **graphy**

ctxt0: **SEESMEEEAUCSRTGPRS**

Encryption:



ctxt: **EAGSERMCRSUPEETESS**

Example: Germany, WWI

- French were able to decrypt after seeing several messages of the same length

Bifid Cipher

Polybius square + Transposition + Inverse Polybius

	1	2	3	4	5
1	y	n	r	b	f
2	d	l	w	o	g
3	s	p	a	t	k
4	h	v	ij	x	c
5	q	u	z	e	m

plaintext: **super secret message**

Polybius: **35351 354153 5533325
12243 145344 5411354**

Transpose: **353513541535533325122431453445411354**

Inv.Polybius:**k k r e f k z a g n o s c t c h r e**

Bifid Cipher

Polybius square + Transposition + Inverse Polybius

Invented in 1901 by Felix Delastelle

Each ctext character depends on two ptxt characters

- Still possible to break using frequency analysis

Next Time

“Pre-modern” Crypto Part II:
Enter technology

Reminders

By Friday Feb 7th:

- HW0: Fill out OH Doodle poll

Homework 1, Project 1 to be released next week

Start looking for project teams

Send me Hollywood Crypto examples!