COS 433/Math 473: Cryptography

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Fall 2020

Announcements/Reminders

- HW1 due September 15
- PR1 due October 6

Previously on COS 433…

Perfect Security for Multiple Messages

Definition: A stateless scheme **(Enc,Dec)** has **perfect secrecy for d messages** if, for any two sequences of messages $(m_0^{(i)})_{i\in[d]}$, $(m_1^{(i)})_{i\in[d]} \in M^d$

$$\left(\mathrm{Enc}(K,\ m_0^{(i)}\ )\right)_{i\in[d]} \overset{d}{=} \left(\mathrm{Enc}(K,\ m_1^{(i)}\ )\right)_{i\in[d]}$$

Notation: $(\ f(i)\ )_{i\in[d]} = (\ f(1),\ f(2),\ \dots,\ f(d)\ )$

Randomized Encryption

Syntax:

- Key space **K** (usually $\{0,1\}^\lambda$)
- Message space **M** (usually $\{0,1\}^n$)
- Ciphertext space **C** (usually $\{0,1\}^m$)
- **Enc: K×M → C** (potentially probabilistic)
- **Dec: K×C → M** (usually deterministic)

Correctness:

- For all $k \in K$, $m \in M$,

$$\Pr[\ \mathrm{Dec}(k, \mathrm{Enc}(k,m)\ ) = m\ ] \quad = \quad 1$$

Theorem: No stateless *randomized* encryption scheme can have perfect security for multiple messages

Proof of Easy Case

Let $\mathbf{(Enc,Dec)}$ be stateless, deterministic

Let $\mathbf{m_0^{(0)} = m_0^{(1)}}$
Let $\mathbf{m_1^{(0)} \neq m_1^{(1)}}$

Consider distributions of encryptions:

- $\mathbf{(c^{(0)} , c^{(1)}) = (Enc(K, m_0^{(0)}), Enc(K, m_0^{(1)}))}$

  $\Rightarrow \mathbf{c^{(0)} = c^{(1)}}$ (by determinism)

- $\mathbf{(c^{(0)} , c^{(1)}) = (Enc(K, m_1^{(0)}), Enc(K, m_1^{(1)}))}$

  $\Rightarrow \mathbf{c^{(0)} \neq c^{(1)}}$ (by correctness)

Generalize to Randomized Encryption

Let **(Enc,Dec)** be stateless, ~~deterministic~~

Let $\mathbf{m_0^{(0)} = m_0^{(1)}}$
Let $\mathbf{m_1^{(0)} \neq m_1^{(1)}}$

Consider distributions of encryptions:

- $\mathbf{(c^{(0)} , c^{(1)}) = (Enc(K, m_0^{(0)}), Enc(K, m_0^{(1)}))}$

  $\Rightarrow$ **????**

- $\mathbf{(c^{(0)} , c^{(1)}) = (Enc(K, m_1^{(0)}), Enc(K, m_1^{(1)}))}$

  $\Rightarrow \mathbf{c^{(0)} \neq c^{(1)}}$ (by correctness)

Generalize to Randomized Encryption

$(c^{(0)} , c^{(1)}) = (\mathrm{Enc}(K, m), \mathrm{Enc}(K, m))$

$\Pr[c^{(0)} = c^{(1)}]$?

- Fix $\mathbf{k}$
- Conditioned on $\mathbf{k}$, ciphertexts $c^{(0)}$ and $c^{(1)}$ are two independent samples from same distribution $\mathrm{Enc}(k, m)$

Lemma: Given any distribution **D** over a finite set **X**, $\Pr[Y=Y' : Y \leftarrow D, Y' \leftarrow D] \geq 1/|X|$

- Therefore, $\Pr[c^{(0)} = c^{(1)}]$ is non-zero

Generalize to Randomized Encryption

Let **(Enc,Dec)** be stateless, deterministic

Let $m_0^{(0)} = m_0^{(1)}$
Let $m_1^{(0)} \neq m_1^{(1)}$

Consider distributions of encryptions:

- $(c^{(0)} , c^{(1)}) = (\mathrm{Enc}(K, m_0^{(0)}), \mathrm{Enc}(K, m_0^{(1)}))$

  $\Rightarrow \Pr[c^{(0)} = c^{(1)}] > 0$

- $(c^{(0)} , c^{(1)}) = (\mathrm{Enc}(K, m_1^{(0)}), \mathrm{Enc}(K, m_1^{(1)}))$

  $\Rightarrow \Pr[c^{(0)} = c^{(1)}] = 0$

Today: Relaxing Perfect Secrecy

What do we do now?

Tolerate tiny probability of distinguishing

- If $\Pr[c^{(0)} = c^{(1)}] = 2^{-128}$, in reality never going to happen

How Small Is Ok?

Practice:

- Something unlikely to happen in lifetime of data/person/civilization/universe

- Typically something like 2^{-80}, 2^{-128}, or maybe 2^{-256}
  - Being struck by lightning twice: 2^{-23}
  - Winning the lottery: 2^{-26}
  - World-ending asteroid while on this slide: 2^{-46}

How Small Is Ok?

Theory:

- Maybe things will change as technology improves
- Want a more conceptual answer
- Absolute constants unsatisfactory
- Instead, use “negligible” functions

Negligible functions

Def: A function $\mathbf{f}$ is **polynomial** if $\mathbf{f(n) = O(n^c)}$ for some constant $\mathbf{c}$

Def: A function $\mathbf{g}$ is **super-polynomial** if, for every polynomial $\mathbf{f}$, $\mathbf{f(n) = O(g(n))}$

Def: A function $\mathbf{p}$ is **inverse polynomial** if $\mathbf{1/p(n)}$ is polynomial

Def: A function $\boldsymbol{\varepsilon}$ is **negligible** if, for every inverse polynomial $\mathbf{p}$, $\boldsymbol{\varepsilon}\mathbf{(n) = O(p(n))}$

(equivalently, $\mathbf{1/}\boldsymbol{\varepsilon}$ is super-polynomial)

Examples

2^n	super-polynomial
$n^{-n/7}$	negligible
$3^{-5\log n}$	inverse polynomial ($= n^{-5\log 3}$)
$1.5^{-\sqrt[3]{n}}$	negligible
$8^{\log^3 n}$	super-polynomial ($= n^{(\log 8)(\log^2 n)}$)
$(\log n)/n$	inverse polynomial

Security Parameter **λ**

Additional input to system, dictates “security level”

Key, message, ciphertext size all **polynomial** in **λ**

Probability of adversary success is **negligible** in **λ**

Defining Encryption Again

Syntax:

- Key space $\mathbf{K_\lambda}$
- Message space $\mathbf{M_\lambda}$
- Ciphertext space $\mathbf{C_\lambda}$
- **Enc:** $\mathbf{K_\lambda \times M_\lambda \rightarrow C_\lambda}$ (potentially randomized)
- **Dec:** $\mathbf{K_\lambda \times C_\lambda \rightarrow M_\lambda}$

Correctness:

- $\mathbf{\log|K_\lambda|, \log|M_\lambda|, \log|C_\lambda|}$ polynomial in $\boldsymbol{\lambda}$
- For all $\boldsymbol{\lambda}$, $\mathbf{k \in K_\lambda,\ m \in M_\lambda,}$

$$\mathbf{Pr[Dec(k, Enc(k,m)) = m] = 1}$$

Statistical Distance

Given two distributions $\mathbf{D_1}$, $\mathbf{D_2}$ over a set $\mathbf{X}$, define

$$\Delta(D_1,D_2) = ½\Sigma_x \mid \Pr[D_1=x] - \Pr[D_2=x] \mid$$

Observations:

$$0 \leq \Delta(D_1,D_2) \leq 1$$

$$\Delta(D_1,D_2) = 0 \iff D_1 \overset{d}{=} D_2$$

$$\Delta(D_1,D_2) \leq \Delta(D_1,D_3) + \Delta(D_3,D_2)$$

(**Δ** is a metric)

Another View of Statistical Distance

Theorem: $\Delta(D_1,D_2) \geq \varepsilon$ iff $\exists$ (potentially randomized) A s.t.

$$| \Pr[A(D_1) = 1] - \Pr[A(D_2) = 1] | \geq \varepsilon$$

Terminology: for any A,

$$| \Pr[A(D_1) = 1] - \Pr[A(D_2) = 1] |$$

is called the "advantage" of A in distinguishing D_1 and D_2

Another View of Statistical Distance

Theorem: $\Delta(D_1,D_2) \geq \varepsilon$ iff $\exists$ (potentially randomized) A s.t.

$$\left| \Pr[A(D_1) = 1] - \Pr[A(D_2) = 1] \right| \geq \varepsilon$$

To lower bound Δ, just need to show adversary A with that advantage

Examples

D_1 = Uniform distribution over $\{0,1\}^n$

- $\Pr[D_1=x] = 2^{-n}$

D_2 = Uniform conditioned on even parity

- $\Pr[D_2=x] = 2^{-(n-1)}$ if x has even parity, 0 otherwise

$$\begin{aligned}\Delta(D_1,D_2) &= \tfrac{1}{2}\Sigma_{\text{even }x}\ |2^{-n} - 2^{-(n-1)}| \\ &\quad + \tfrac{1}{2}\Sigma_{\text{odd }x}\ |2^{-n} - 0| \\ &= \tfrac{1}{2}\Sigma_{\text{even }x}\ 2^{-n} + \tfrac{1}{2}\Sigma_{\text{odd }x}\ 2^{-n} \\ &= \tfrac{1}{2}\end{aligned}$$

Examples

$D_1 =$ Uniform over $\{1,\ldots,n\}$

$D_2 =$ Uniform over $\{1,\ldots,n+1\}$

$$\Delta(D_1,D_2) = \tfrac{1}{2}\sum_{x=1}^{n} |1/n - 1/(n+1)| + \tfrac{1}{2}\,|0 - 1/(n+1)|$$

$$= \tfrac{1}{2}\sum_{x=1}^{n} 1/n(n+1) \;+\; \tfrac{1}{2}\,1/(n+1)$$

$$= \tfrac{1}{2}\,1/(n+1) + \tfrac{1}{2}\,1/(n+1) = 1/(n+1)$$

Statistical Security (Concrete)

Definition: A scheme **(Enc,Dec)** has **ε-statistical secrecy for d messages** if $\forall$ two sequences of messages $(m_0^{(i)})_{i\in[d]}$, $(m_1^{(i)})_{i\in[d]} \in M^d$

$$\Delta\left[\ \left(\mathrm{Enc}(K,\ m_0^{(i)}\ )\right)_{i\in[d]},\quad \left(\mathrm{Enc}(K,\ m_1^{(i)}\ )\right)_{i\in[d]}\ \right] < \varepsilon$$

We will call such a scheme **(d,ε)** statistically secure

Statistical Security (Asymptotic)

Definition: A scheme **(Enc,Dec)** has **statistical secrecy for d messages** if $\exists$ negligible **ε** such that $\forall$ two sequences $(m_0^{(i)})_{i\in[d]}$, $(m_1^{(i)})_{i\in[d]}$ $\in$ M_λ^d,

$$\Delta\Big[\ \big(\mathrm{Enc}(K_\lambda,\ m_0^{(i)}\ )\big)_{i\in[d]},$$

$$\big(\mathrm{Enc}(K_\lambda,\ m_1^{(i)}\ )\big)_{i\in[d]}\ \Big] < \varepsilon(\lambda)$$

We will call such a scheme **d**-time statistically secure

Stateless Encryption with Multiple Messages

Ex:

$M = C$
$K = \text{Perms}(M)$
$\text{Enc}(K, m) = K(m)$
$\text{Dec}(K, c) = K^{-1}(c)$

Q: Is this statistically secure for two messages?

Theorem: For any **ε<1**, no stateless *deterministic* encryption scheme can have **ε**-statistical security for **2** messages

(Proof basically the same as before)

Importantly: proof does **not** hold for randomized schemes for **ε>0**

Stateless Encryption with Multiple Messages

Ex:

$r \leftarrow R$

$C = M \times R$
$K = \text{Perms}(C)$
$\text{Enc}(K, m) = K(m, r)$
$\text{Dec}(K, c) = (m', r') \leftarrow K^{-1}(c)$, output m'

Q: Is this statistically secure for two messages?

Q: Is it practical?

Example

A more efficient example:

$M = \mathbb{Z}_p$ (p a prime of size 2^λ, $\lambda=128$)

$C = \mathbb{Z}_p{}^2$

$K = \mathbb{Z}_p{}^2$

Random in $\mathbb{Z}_p$

$\mathrm{Enc}(\ (a,b),\ m) = (r,\ \ (ar+b) + m\ )$

$\mathrm{Dec}(\ (a,b),\ (r,c)\ ) = c - (ar+b)$

Proof of Example

Let $\mathbf{D_b}$ be distribution of $\mathbf{(\ Enc(k,m_b^{(i)})\ )_{i\in\{0,1\}}}$
Let $\mathbf{D_b'}$ be the following:

1. Run $\mathbf{(c_0,c_1)\leftarrow Db}$
2. If $\mathbf{r_0=r_1}$, output $\perp$
3. Else output $\mathbf{(c_0,c_1)}$

Fix $\mathbf{r_0\neq r_1,\ m_0,m_1,c_0,c_1}$

$$\Pr_{(a,b)}[ar_0+b+m_0=c_0,\ ar_1+b+m_1=c_1] = 1/p^2$$

So $\mathbf{D_0' \overset{d}{=} D_1'}$ $\mathbf{(\ \Delta(D_0',\ D_1') = 0\ )}$

The Symbol ⊥ (“bot”)

Represents an abort/reject/bad outcome

Augments whatever set we are talking about

- Ex: support of $\mathbf{D_b = C^2}$, so support of $\mathbf{D_b' = C^2} \cup \mathbf{\{\perp\}}$

Proof of Example

Lemma: $\Delta(D_1,D_2) \leq \frac{1}{2}\Pr[\text{bad}|D_1] + \frac{1}{2}\Pr[\text{bad}|D_2] + \Delta(D_1' , D_2')$

Where:

- "**bad**" is some event
- $\Pr[\text{bad}|D_b]$ is probability "**bad**" when sampling from D_b
- D_b' is D_b, except outputs $\perp$ on "**bad**"

Proof of Lemma

$$\Delta(D_1,D_2) = \tfrac{1}{2}\Sigma_x|\ \Pr[D_1=x] - \Pr[D_2=x]\ |$$

$$= \tfrac{1}{2}\Sigma_{x:bad}|\ \Pr[D_1=x\ ] - \Pr[D_2=x\ ]\ |$$
$$+ \tfrac{1}{2}\Sigma_{x:good}|\ \Pr[D_1=x\ ] - \Pr[D_2=x\ ]\ |$$

$$\leq \tfrac{1}{2}\Sigma_{x:bad}|\ \Pr[D_1=x\ ]\ | + \tfrac{1}{2}\Sigma_{x:bad}|\ \Pr[D_2=x\ ]\ |$$
$$+ \tfrac{1}{2}\Sigma_{x:good}|\ \Pr[D_1=x] - \Pr[D_2=x]\ |$$

$$= \tfrac{1}{2}\Sigma_{x:bad}|\ \Pr[D_1=x\ ]\ | + \tfrac{1}{2}\Sigma_{x:bad}|\ \Pr[D_2=x\ ]\ |$$
$$+ \tfrac{1}{2}\Sigma_x|\ \Pr[D_1'=x] - \Pr[D_2'=x]\ |$$

$$= \tfrac{1}{2}\ \Pr[bad|D_1] + \tfrac{1}{2}\ \Pr[bad|D_2] + \Delta(D_1',D_2')$$

Proof of Example

Let $\mathbf{D_b}$ be distribution of $\mathbf{(\ \ Enc(k,m_b^{(i)})\ \ )_{i\in\{0,1\}}}$
Let **bad** be when $\mathbf{r_0=r_1}$
Let $\mathbf{D_b}$' be the following:

1. Run $\mathbf{(c_0,c_1)\leftarrow Db}$
2. If **bad**, output ⊥
3. Else output $\mathbf{(c_0,c_1)}$

$\mathbf{Pr[bad|D_b] = 1/p}$
$\mathbf{\Delta(D_0', D_1') = 0}$

Therefore, $\mathbf{\Delta(D_0, D_1) \leq 1/p \approx 2^{-\lambda}}$

Summary so Far

Stateless encryption for multiple messages ✓

But, key length grows with number of messages ✘

And, key length grows with length of message ✘

Limits of Statistical Security

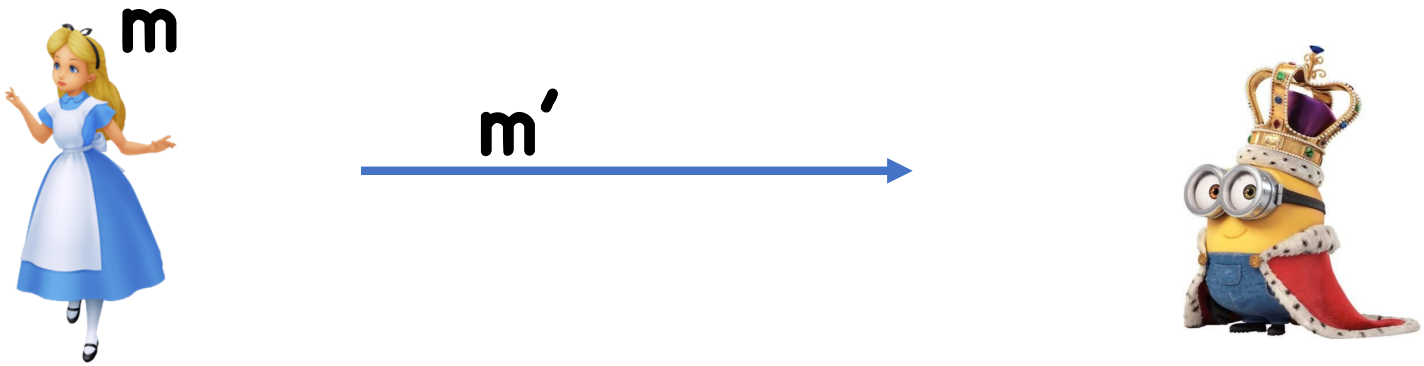
Theorem: Suppose **(Enc,Dec)** has plaintext space $\mathbf{M} = \{0,1\}^n$ and key space $\mathbf{K} = \{0,1\}^t$. Moreover, assume it is **(d, 0.4999)**-secure. Then:

$$t \geq d\ n$$

In other words, the key must be at least as long as the total length of all messages encrypted

Proof Idea: Compression

Use an encryption protocol to build a compression protocol



m

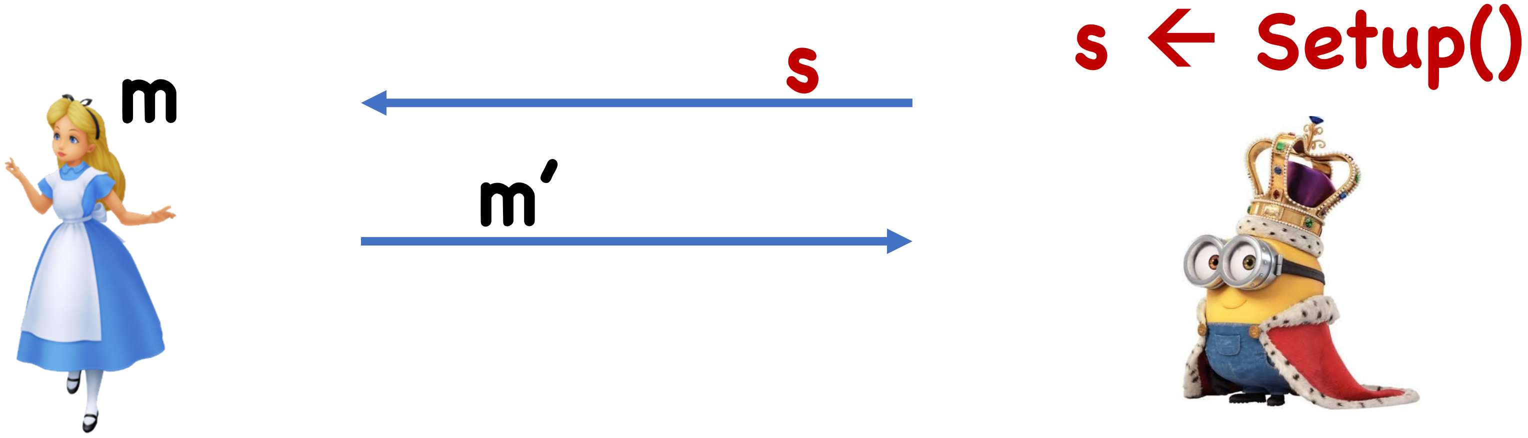
m'

$m' \leftarrow \text{Comp}(m)$

$m \leftarrow \text{Decomp}(m')$

Goal: $|m'| < |m|$

For Now: Easier Goal



m

s ← Setup()

s

m′

m′ ← Comp(s,m)

m ← Decomp(s,m′)

Goal: $|m'| < |m|$

The Protocol

Let $\mathbf{m_0}$ be some arbitrary message in $\mathbf{M}$

Setup():

- Choose random $\mathbf{k_0 \leftarrow K}$
- **Let** $\mathbf{c_1 \leftarrow Enc(k_0,m_0), \ldots, c_d \leftarrow Enc(k_0,m_0)}$
- **Output** $\mathbf{(c_1,\ldots,c_d)}$

In $\mathbf{M^d}$

Comp($\mathbf{(c_1,\ldots,c_d), (m_1,\ldots,m_d)}$ **):**

- Find $\mathbf{k,r_1,\ldots,r_d}$ such that $\mathbf{c_i=Enc(k,m_i;\ r_i)}\ \ \forall \mathbf{i}$
- If no such values exist, abort
- Output $\mathbf{k}$

The Protocol

Let $\mathbf{m_0}$ be some message in $\mathbf{M}$

In $\mathbf{M}^d$

$\mathbf{Comp(\ (c_1,\ldots,c_d),\ (m_1,\ldots,m_d)\ ):}$
- Find $\mathbf{k,r_1,\ldots,r_d}$ such that $\mathbf{c_i{=}Enc(k,m_i;\ r_i)}\ \forall\,\mathbf{i}$
- If no such values exist, abort
- Output $\mathbf{k}$

$\mathbf{Decomp((c_1,\ldots,c_d),\ k\ ):}$
- Compute $\mathbf{m_i = Dec(k,c_i)}$
- Output $\mathbf{(m_1,\ldots,m_d)}$

Analysis of Protocol

If **Comp** succeeds, **Decomp** must succeed by correctness

- Since $\mathbf{c_i{=}Enc(k,m_i;\ r_i)}$, $\mathbf{Dec(k,c_i)}$ must give $\mathbf{m_i}$

Therefore, must figure out when **Comp** succeeds

Claim: For any sequence of messages $\mathbf{m_1,\ldots,m_d}$, **Comp** succeeds with probability at least $\mathbf{1-\varepsilon}$

(Probability over the randomness used by **Setup()**)

Claim: For any sequence of messages $m_1,\ldots,m_d$, **Comp** succeeds with probability at least $1-\varepsilon$

Proof:

- Suppose **Comp** succeeds with probability $1-p$ for messages $m_1,\ldots,m_d$
- Let $A(c_1,\ldots,c_d)$ be the algorithm that runs $\mathrm{Comp}((c_1,\ldots,c_d),\ (m_1,\ldots,m_d))$ and outputs **1** if **Comp** succeeds

- If $c_i = \mathrm{Enc}(k_0,m_i)$, then $\Pr[A(c_1,\ldots,c_d)=1] = 1$
- If $c_i = \mathrm{Enc}(k_0,m_0)$, then $\Pr[A(c_1,\ldots,c_d)=1] = 1-p$

- By (d,ε)-statistical security of **Enc**, p must be $\leq\varepsilon$

Claim: For any sequence of messages $\mathbf{m_1,\ldots,m_d}$, **Comp** succeeds with probability at least $\mathbf{1-\varepsilon}$

Claim: For **a random** sequence of messages $\mathbf{m_1,\ldots,m_d}$, **Comp** succeeds with prob at least $\mathbf{1-\varepsilon}$

(Probability over the randomness used by **Setup()** and the random choices of $\mathbf{m_1,\ldots,m_d}$)

Next step: Removing Setup

We know:

$$\Pr\left[\text{Comp succeeds}: \begin{array}{l}(c_1,\ldots,c_d)\leftarrow\text{Setup}(),\\ m_i\leftarrow M\end{array}\right] \geq 1-\varepsilon$$

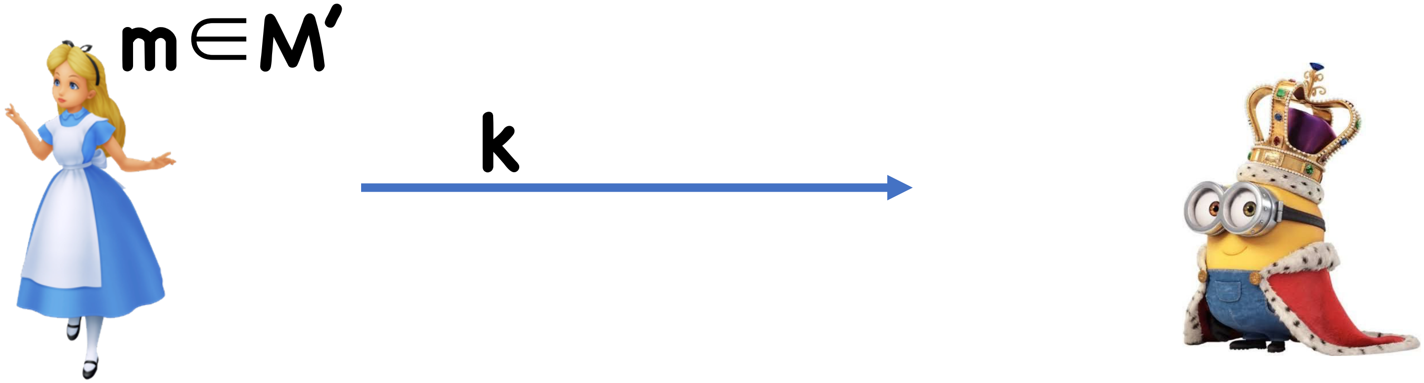
$\Longrightarrow$ there must exist *some* $(c_1^*,\ldots,c_d^*)$ such that

$$\Pr[\text{Comp succeeds}: m_i\leftarrow M] \geq 1-\varepsilon$$

Fix $(c_1^*,\ldots,c_d^*)$, define: $M' = \{(m_1,\ldots,m_d)$: Comp succeeds$\}$

- Note that $|M'| \geq (1-\varepsilon)\ |M|^d$

The Protocol



Find $k,r_1,\dots,r_d$ such that
$c_i^*=\mathrm{Enc}(k,m_{i;}\ r_i)\ \forall i$

For each i,

Let $m_i\leftarrow\mathrm{Dec}(k,c_i^*)$

Output $(m_1,\dots,m_d)$

By previous analysis,

- Alice always successfully compresses
- Bob always successfully decompresses

Final Touches

Can compress messages in $\mathbf{M'}$ into keys in $\mathbf{K}$

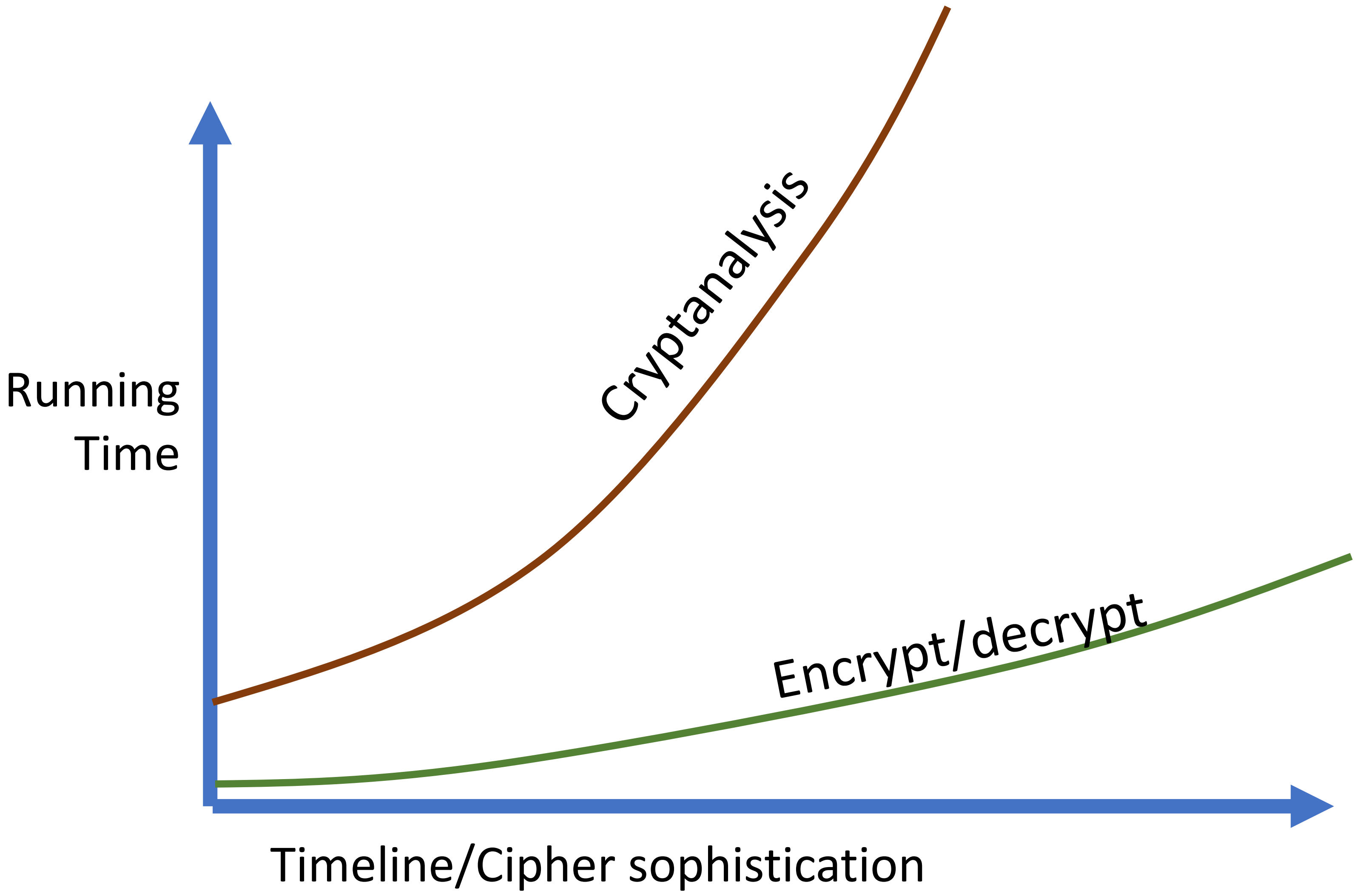
Therefore, it must be that $|\mathbf{M'}| \leq |\mathbf{K}|$

Meaning
$$\begin{aligned} t &= \log |K| \\ &\geq \log |M'| \\ &\geq \log [\ (1-\varepsilon)\ |M|^d\ ] \\ &= d \log |M| + \log [1-\varepsilon] \\ &= dn + \log [1-\varepsilon] \\ &\geq dn \ \text{(as long as } \varepsilon<1/2\text{)} \end{aligned}$$

Takeaway

If you don’t want to physically exchange keys frequently, you cannot obtain statistical security

So, now what?


Running Time
Cryptanalysis
Encrypt/decrypt
Timeline/Cipher sophistication

Computational Security

We are ok if adversary takes a really long time

Only considered attack for adversaries that don’t take too long

How Long Is Ok?

Practice:

- Lifetime of data/person/civilization/universe

- Typically something like 2^{80}, 2^{128}, or maybe 2^{256}
  - Lifetime of universe in nanoseconds: 2^{58}
  - Number of atoms in known universe: 2^{265}

How Long Is Ok?

Theory:

- Maybe things will change as technology improves
- Want a more conceptual answer
- Absolute constants unsatisfactory
- Instead, consider an attack if time bounded by polynomial function

Brute Force Attacks

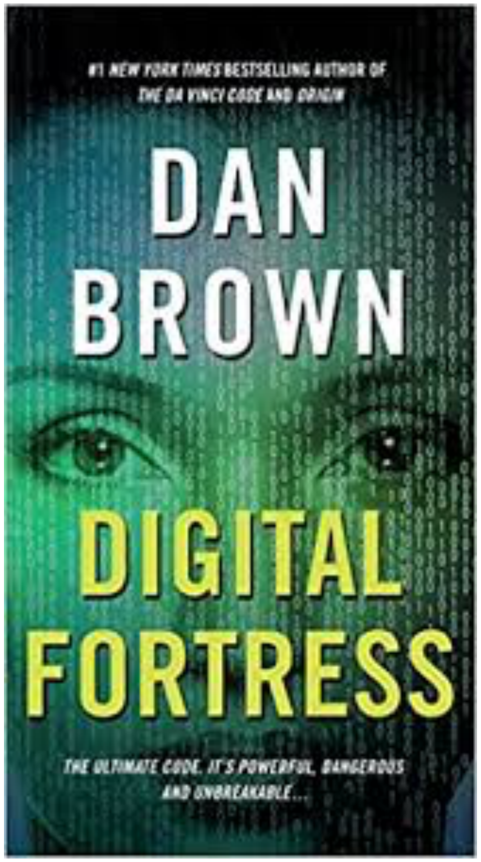
Simply try every key until find right one

If keys have length **λ**, **2^λ** is upper bound on attack

Not always applicable – requires being able to test when guess was correct

- Always applicable when **$|\text{key}| \leq |\text{message}|$**

Holiwudd Criptoe!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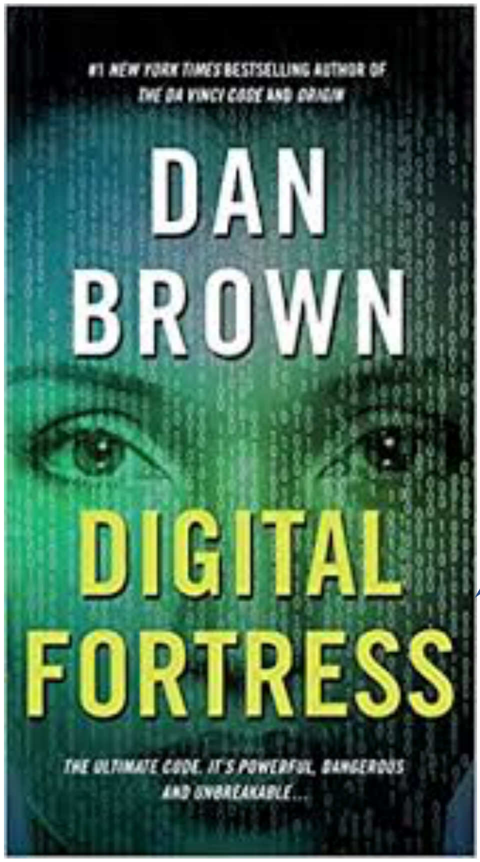
THE ULTIMATE CODE. IT'S POWERFUL, DANGEROUS
AND UNBREAKABLE...

[TRANSLTR]’s three million processors would all work in parallel ... trying every new permutation as they went

Holiwudd Criptoe!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“What’s the longest you’ve ever seen TRANSLTR take to break a code?”

“About an hour, but it had a ridiculously long key—ten thousand bits”

Reminders

- HW1 due September 15
- PR1 due October 6