

COS 433/Math 473: Cryptography

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Fall 2020

Reminders

- Fill out OH poll
- HW1, PR1 to be released today or tomorrow

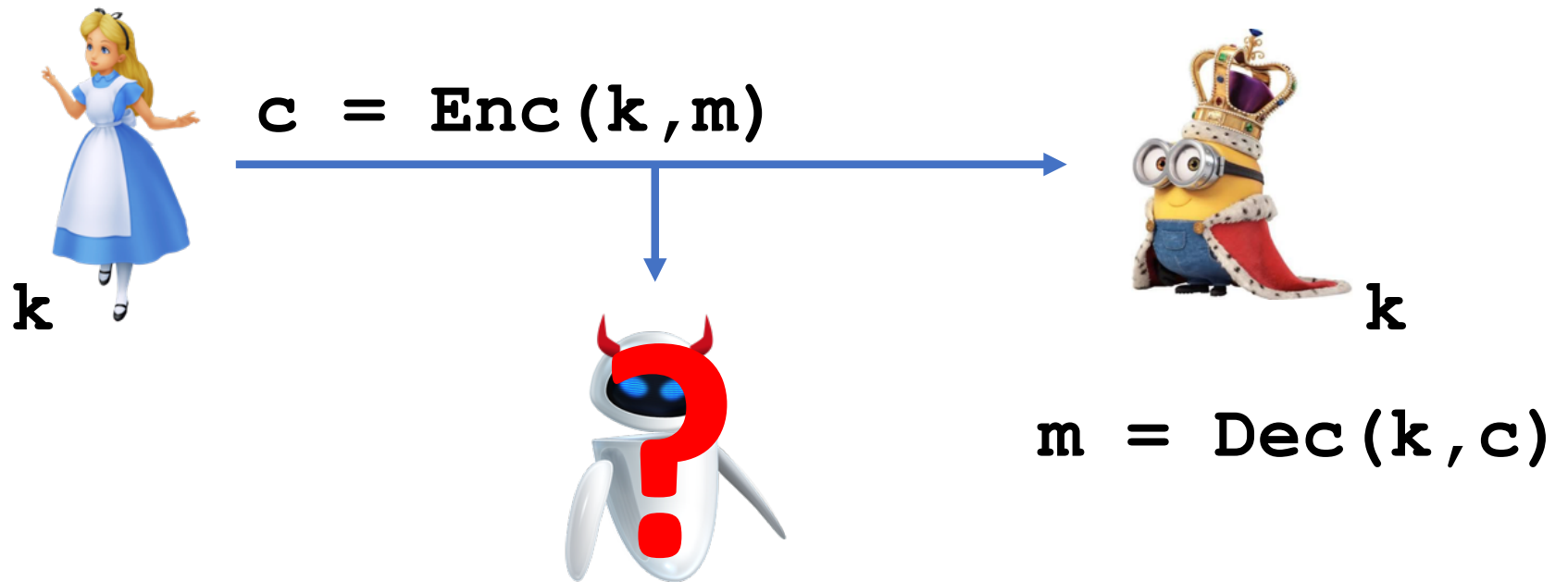
Find teams for projects (up to 4)

Previously on COS 433...

Pre-modern Cryptography

1900 B.C. – mid 1900's A.D

With few exceptions, synonymous with **encryption**



Substitution Ciphers

Apply fixed permutation to plaintext letters

a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
F	M	S	G	Y	U	J	B	T	P	Z	K	E	W	L	Q	H	V	A	X	R	D	N	C	I	O

Example:

plaintext: **super secret message**

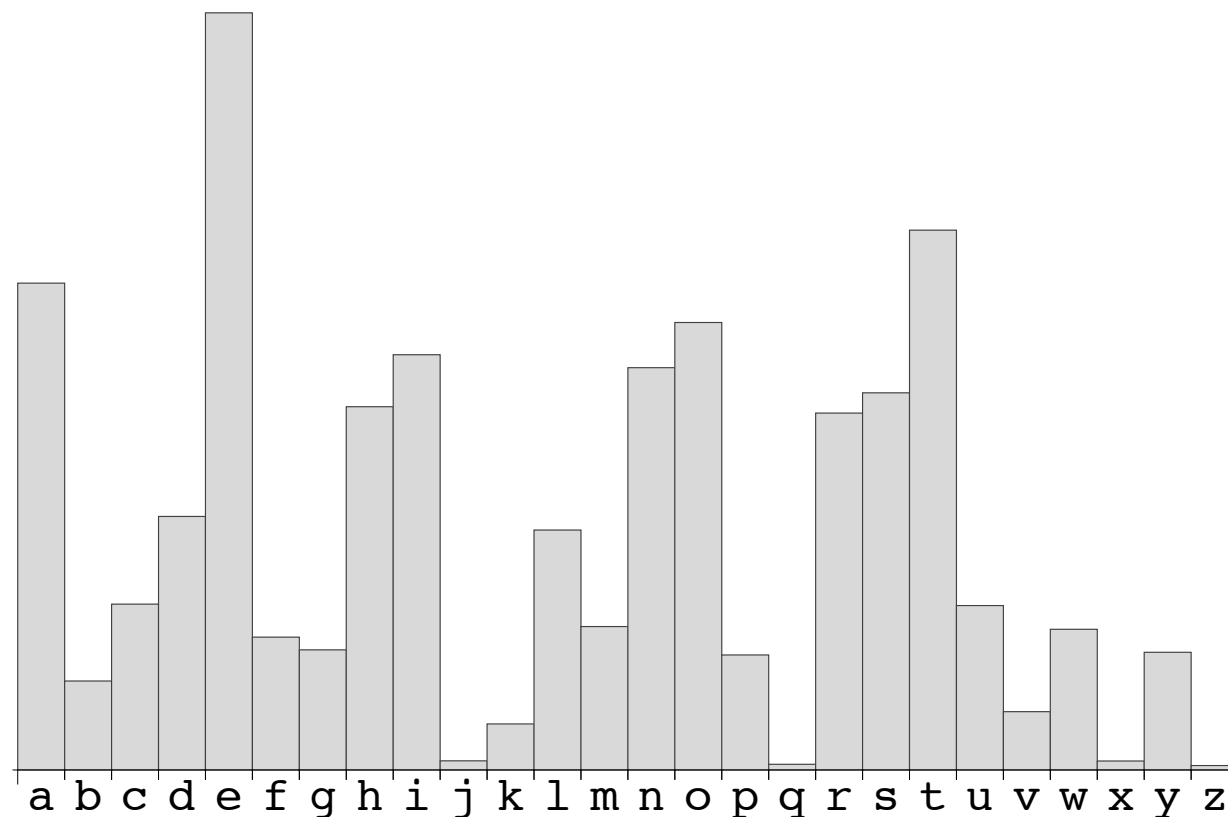
ciphertext: **ARQYV AYSVYX EYAAFJY**

Number of possible keys?

$26! \approx 2^{88}$ ➡ brute force attack expensive

800's A.D. – First Cryptanalysis

Al-Kindi – Frequency Analysis: some characters are more common than others



Polygraphic Substitution

Frequency analysis requires seeing many copies of the same character/group of characters

Idea: encode **d = 2, 3, 4**, etc characters at a time

- New alphabet size: **26^d**
- Symbol frequency decreases:
 - Most common digram: "th", 3.9%
 - trigram: "the", 3.5%
 - quadrigram: "that", 0.8%
- Require much larger ciphertext to perform frequency analysis

Homophonic Substitution

Ciphertexts use a larger alphabet

Common letters have multiple encodings

To encrypt, choose encoding at random

plaintext: **super secret message**

ciphertext: **EKPH9 O3MJ3Z VAOEDNH**

[illegible]

Polyalphabetic Substitution

Use a different substitution for each position

Example: Vigenère cipher

- Sequence of shift ciphers defined by keyword

keyword:	crypt	ocrypt	ocrypto
plaintext:	super	secret	message
ciphertext:	ULNTK	GGTPTM	AGJQPZS

The One-Time Pad

Vigenère on steroids

- Every character gets independent substitution
- Only use key to encrypt one message,
key length \geq message length

keyword:	agule	melpqw	gnspemr
plaintext:	super	secret	message
ciphertext:	SAIPV	EINGUP	SRKHESR

No substitution used more than once, so frequency analysis is impossible

Transposition Ciphers

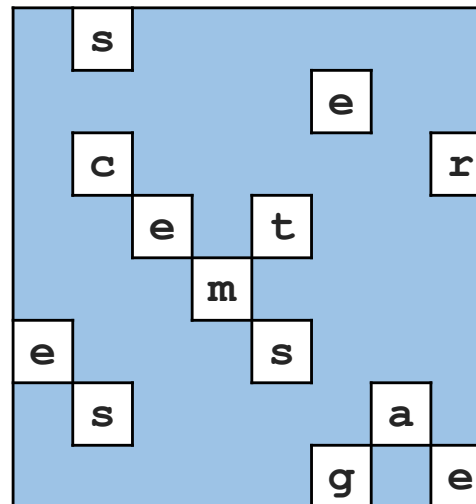
Shuffle plaintext characters

Greek Scytal (600's B.C.)



<https://commons.wikimedia.org/wiki/File:Skytale.png>

Grille (1500's A.D.)



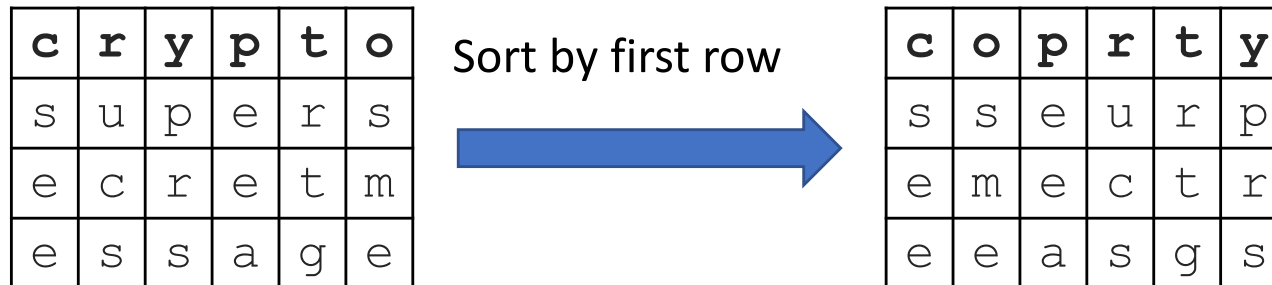
a	s	h	o	e	v	q	k
g	i	p	c	e	e	f	j
e	c	n	i	d	z	w	r
g	i	e	b	t	e	b	o
k	c	d	m	i	z	d	p
e	b	i	d	s	h	e	r
n	s	d	u	r	e	a	v
h	k	e	g	u	g	a	e

Column Transposition

key: **crypto**

ptxt: **supersecretmessage**

Encryption:



ctxt: **SEESMEEEAUCSRTGPRS** (read off columns)

Cryptanalysis:

- Guess key length, reconstruct table
- Look for anagrams in the rows

Today

“Pre-modern” Crypto Part II:
Enter Technology

Disk-based Substitution Ciphers

First Invented by Alberti, 1467



*



†



‡

* cropped from <http://www.cryptomuseum.com/crypto/usa/ccd/img/301058/000/full.jpg>

† cropped from <https://www.flickr.com/photos/austinmills/13430514/sizes/l>

‡ <https://commons.wikimedia.org/wiki/File:Captain-midnight-decoder.jpg>

Disk-based Substitution Ciphers

In most basic form, simple monoalphabetic cipher

Alberti Cipher – rotate the disk periodically

- Considered the first polyalphabetic cipher

Jefferson disk: used by US military until WWII



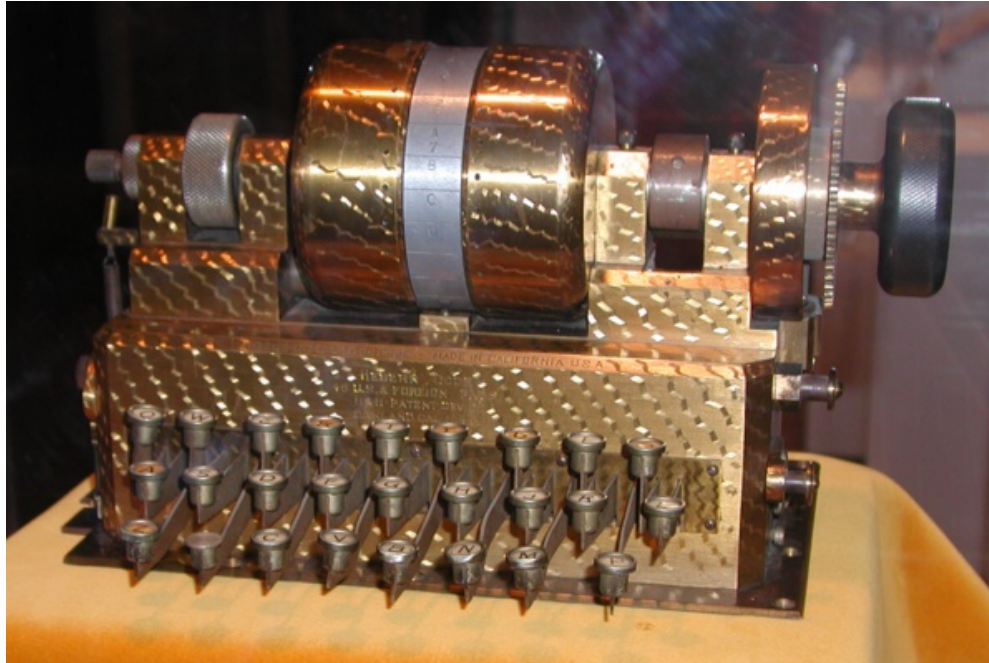
Rotor Machines

Widespread starting in the 1920's

Automatically advance rotor in regular intervals

- Automate process of rotating disk to change substitution
- Eventually allow for more complex substitutions

Rotor Machines



<https://commons.wikimedia.org/wiki/File:Hebern1.jpg>

Rotor contains substitution, advances by one after each stroke, creating different substitution

Rotor Machines

More rotors!



http://americanhistory.si.edu/collections/search/object/nmah_694514

Every time one rotor completes a revolution, it advances the next rotor

Cryptanalysis of Rotor Machines?

d rotors \rightarrow polyalphabetic cipher with key length **26^d**

Possible to break via brute force if only a few rotors

But what if you don't know the permutation given by the rotors?

Edward Hebern vs. William Friedman

Hebern invented machines using 1 to 5 rotors

Tried to sell to US Military, but rejected

Unknown to Hebern, US cryptanalyst Friedman had shown to break the machine, given just **10 ciphertexts**

- And, Friedman wasn't even given rotor wirings!

PURPLE

Diplomatic cipher used by Japanese Foreign Office

Using knowledge gained from cryptanalyzing Hebern's machine, US Intelligence was able to completely reconstruct the cipher machine **using only intercepted ciphertexts**

Friedman's technique applies to essentially any cipher-based machine where fast rotor at one end

Determining Rotor Wirings

Each rotor represents a permutation $\mathbf{R}_1, \mathbf{R}_2, \dots$ on \mathbb{Z}_{26}

If rotor \mathbf{i} has rotated \mathbf{j} times, then it applies the permutation

$$\mathbf{C}^{\mathbf{j}} \circ \mathbf{R}_{\mathbf{i}} \circ \mathbf{C}^{-\mathbf{j}}$$

Where \mathbf{C} maps “a” to “b”, “b” to “c”, etc

Overall permutation:

$$\mathbf{C}^{\mathbf{l}} \circ \mathbf{R}_3 \circ \mathbf{C}^{-\mathbf{l}} \circ \mathbf{C}^{\mathbf{k}} \circ \mathbf{R}_2 \circ \mathbf{C}^{-\mathbf{k}} \circ \mathbf{C}^{\mathbf{j}} \circ \mathbf{R}_1 \circ \mathbf{C}^{-\mathbf{j}}$$

Determining Rotor Wirings

For first 26 letters, only first rotor ever turns

Can write permutation as

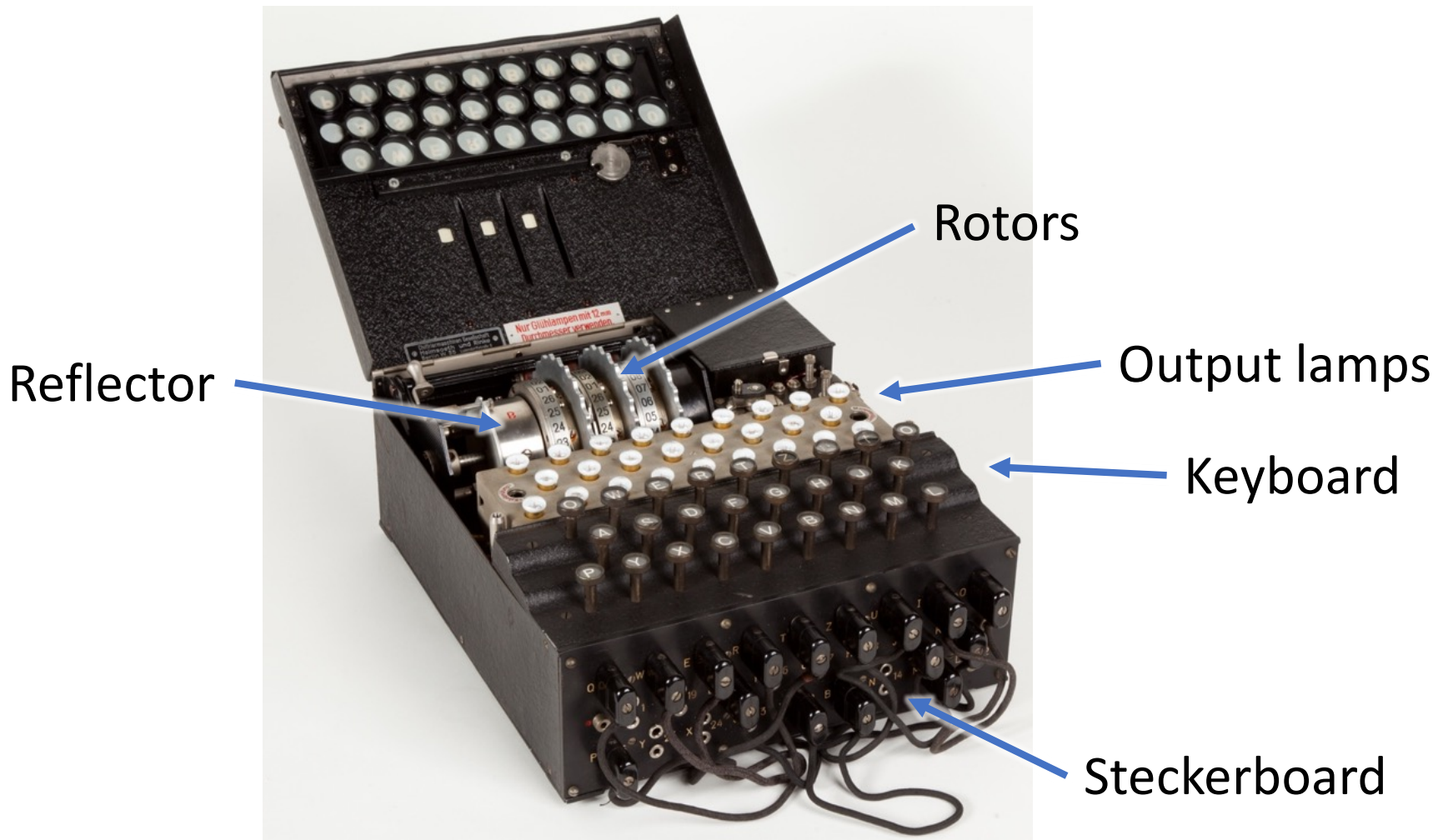
$$\mathbf{L} \circ \mathbf{C}^j \circ \mathbf{R}_1 \circ \mathbf{C}^{-j}$$

For next 26 letters, identical, except different **L**:

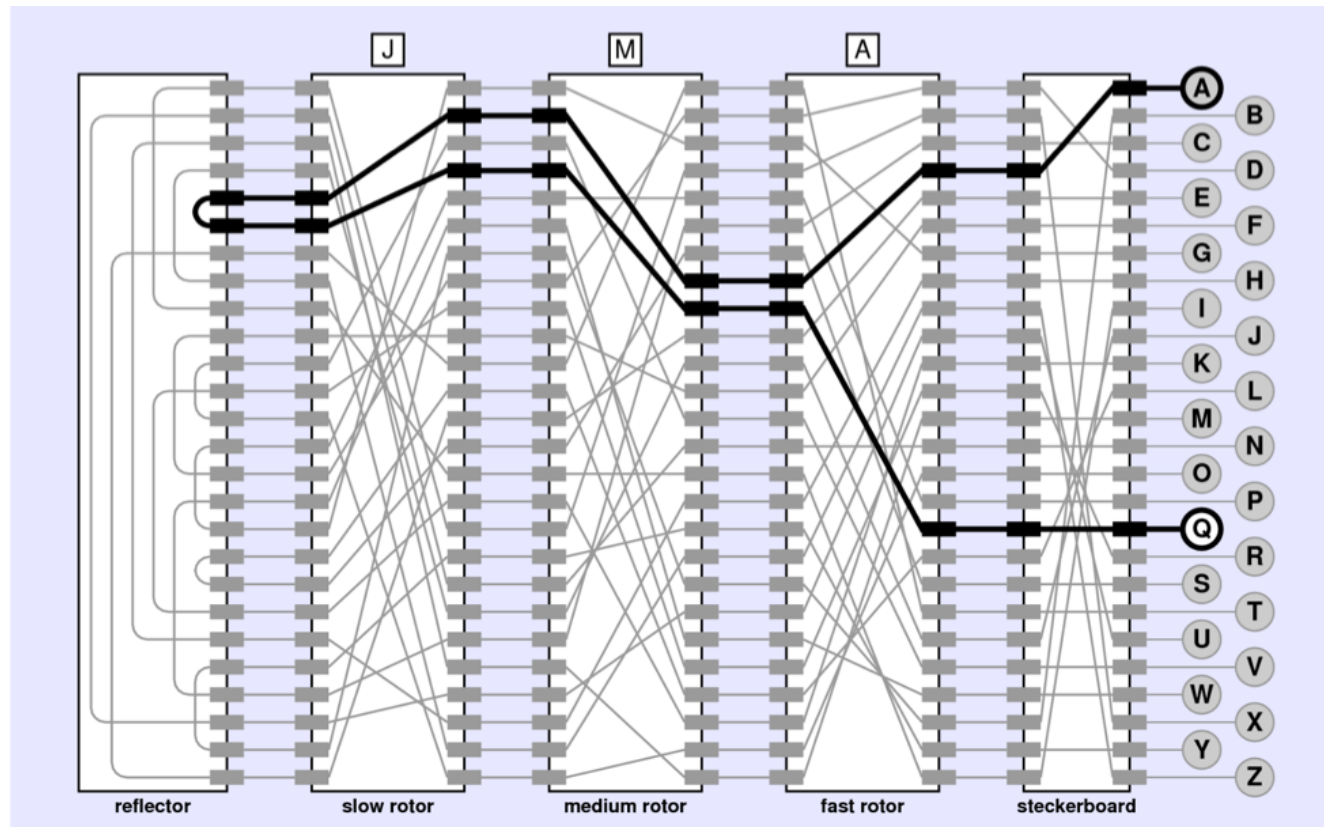
$$\mathbf{L}' \circ \mathbf{C}^j \circ \mathbf{R}_1 \circ \mathbf{C}^{-j}$$

A lot of structure in cipher to exploit

The German Enigma Machine



Enigma Diagram



<http://stanford.edu/class/archive/cs/cs106a/cs106a.1164/handouts/29A-CryptographyChapter.pdf>

Enigma Keys

Key:

- Selection of 3 rotors out of 5 (60 possibilities)
- Initial rotor setting (26^3)
- Steckerboard wiring (216,751,064,975,576)

Possible attack strategies?

- Brute force
 - $\sim 2^{68}$ possible keys: feasible today, but not in WWII
- Frequency analysis
 - Polyalphabetic with key length $26^3 = 17576$
 - Likely no key was used to encrypt enough material

Cracking the Enigma

Key Factors:

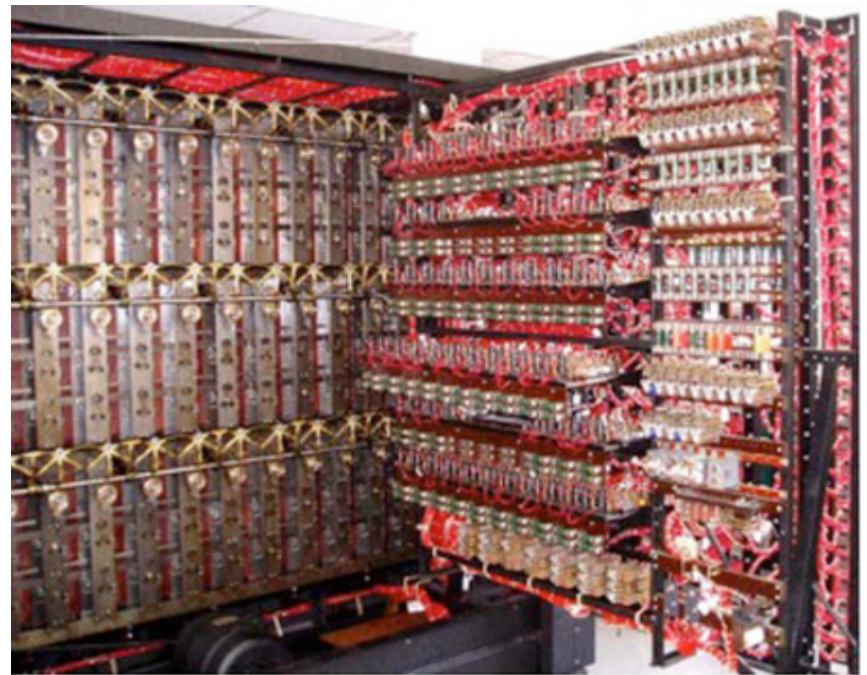
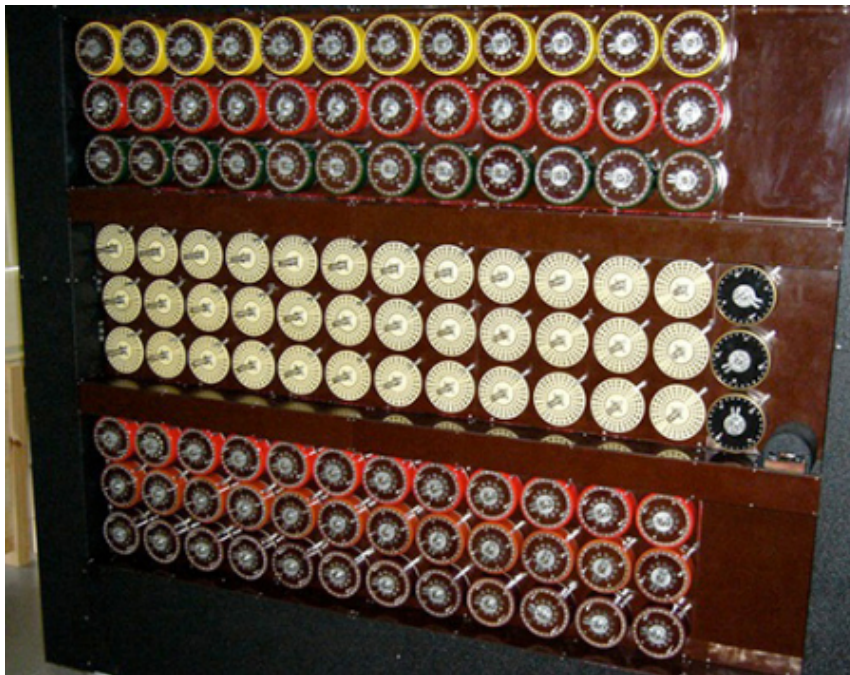
- Captured Enigma device



Cracking the Enigma

Key Factors:

- Technology



Cracking the Enigma

Key Factors:

- User error/bad practices

Hybrid Encryption:

fresh \mathbf{k}' chosen randomly for each ciphertext

$$\mathbf{c} = \text{Enc}(\mathbf{k}, \mathbf{k}') , \text{Enc}(\mathbf{k}' , \mathbf{m})$$

Good: (Now) lots of theory to support this use

Bad: Users would pick bad \mathbf{k}'

Cracking the Enigma

Key Factors:

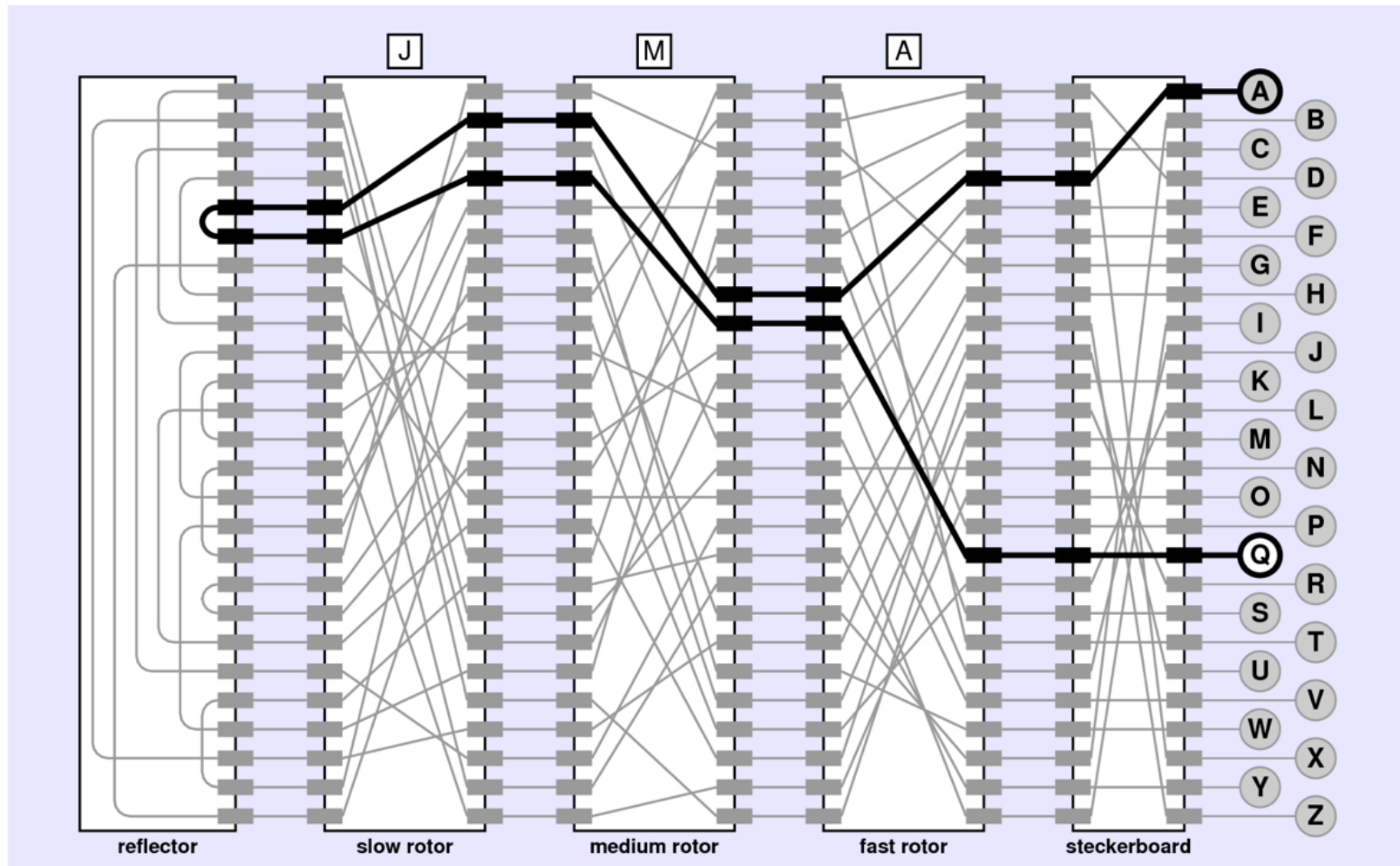
- Known/chosen plaintexts



Cracking the Enigma

Key Factors:

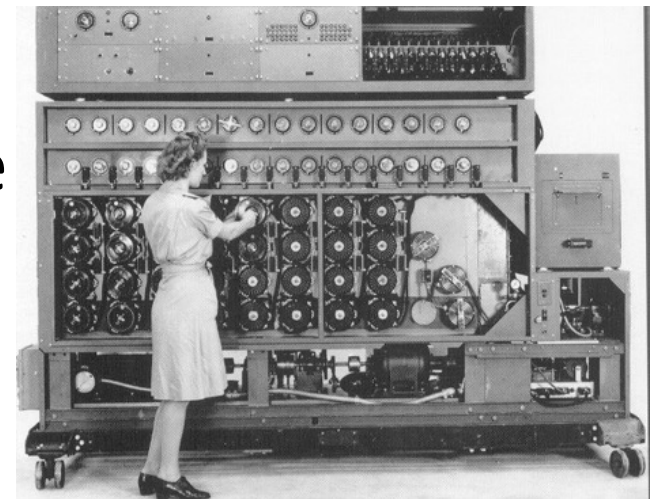
- Mathematical weaknesses



A Key Insight: Loops



- Loops unaffected by steckerboard wiring
- Only need to search the $\approx 2^{20}$ rotor positions to find one that generates such a loop
- Possible at the time using the Bombe



Takeaway: Crypto is Hard

Designing crypto is hard, even experts get it wrong

- Just because I don't know how to break it doesn't mean someone else can't

Unexpected attack vectors

- Known/chosen plaintext attack
- Chosen *ciphertext* attack
- Timing attack
- Power analysis
- Acoustic cryptanalysis

Takeaway: Crypto is Hard

Don't design your own crypto

- You'll probably get it wrong
- Use peer-reviewed schemes instead

Actually, don't even implement your own crypto

- Instead, use well studied crypto library built and tested by many experts

Takeaway: Need for Formalism

For most of history, cipher design and usage based largely on intuition

- Intuition in many cases false

Instead, need to formally define the usage scenario

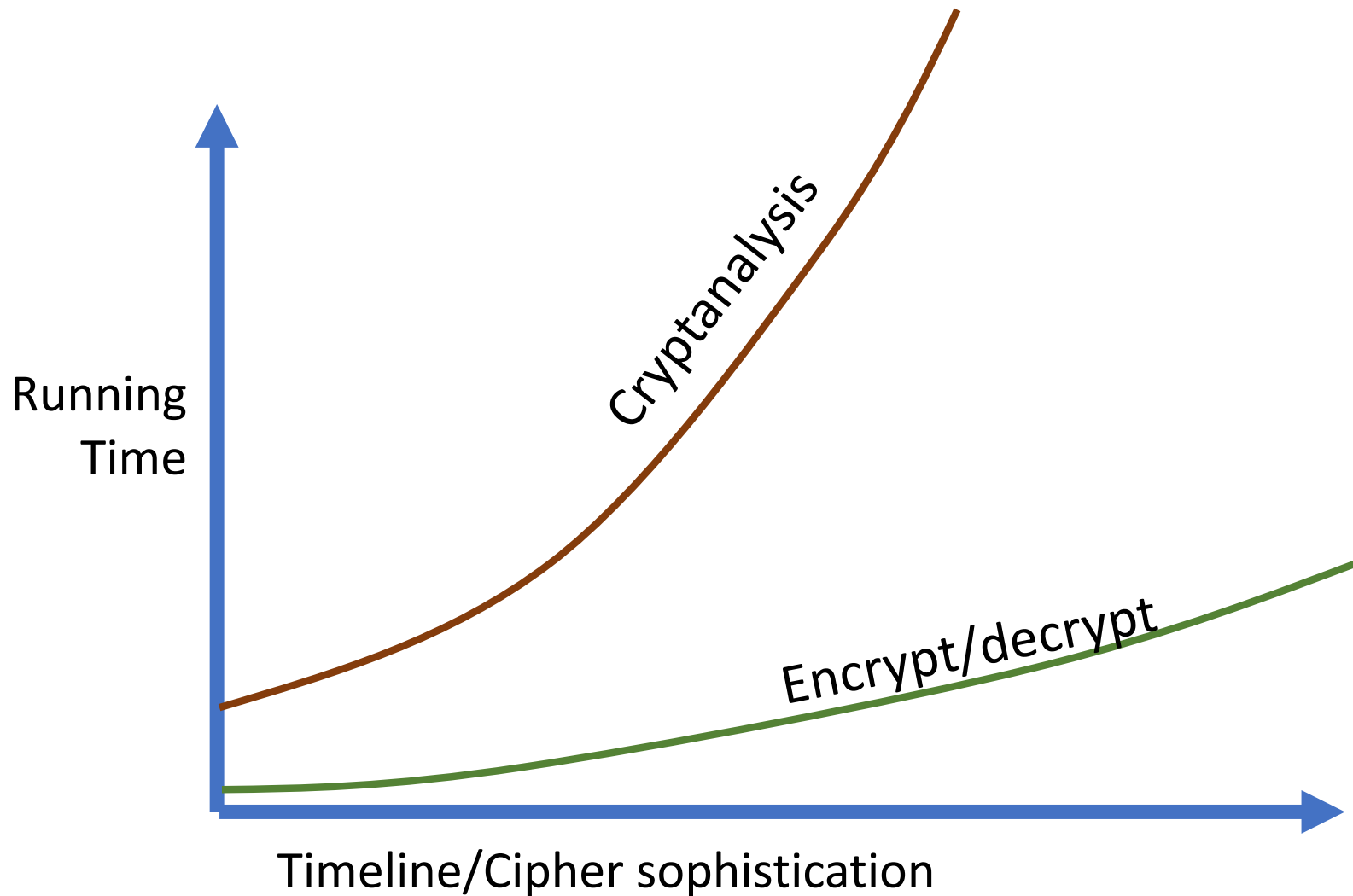
- Prove that scheme is secure in scenario
- Only use scheme in that scenario

Takeaway: Kerckhoffs's Principle

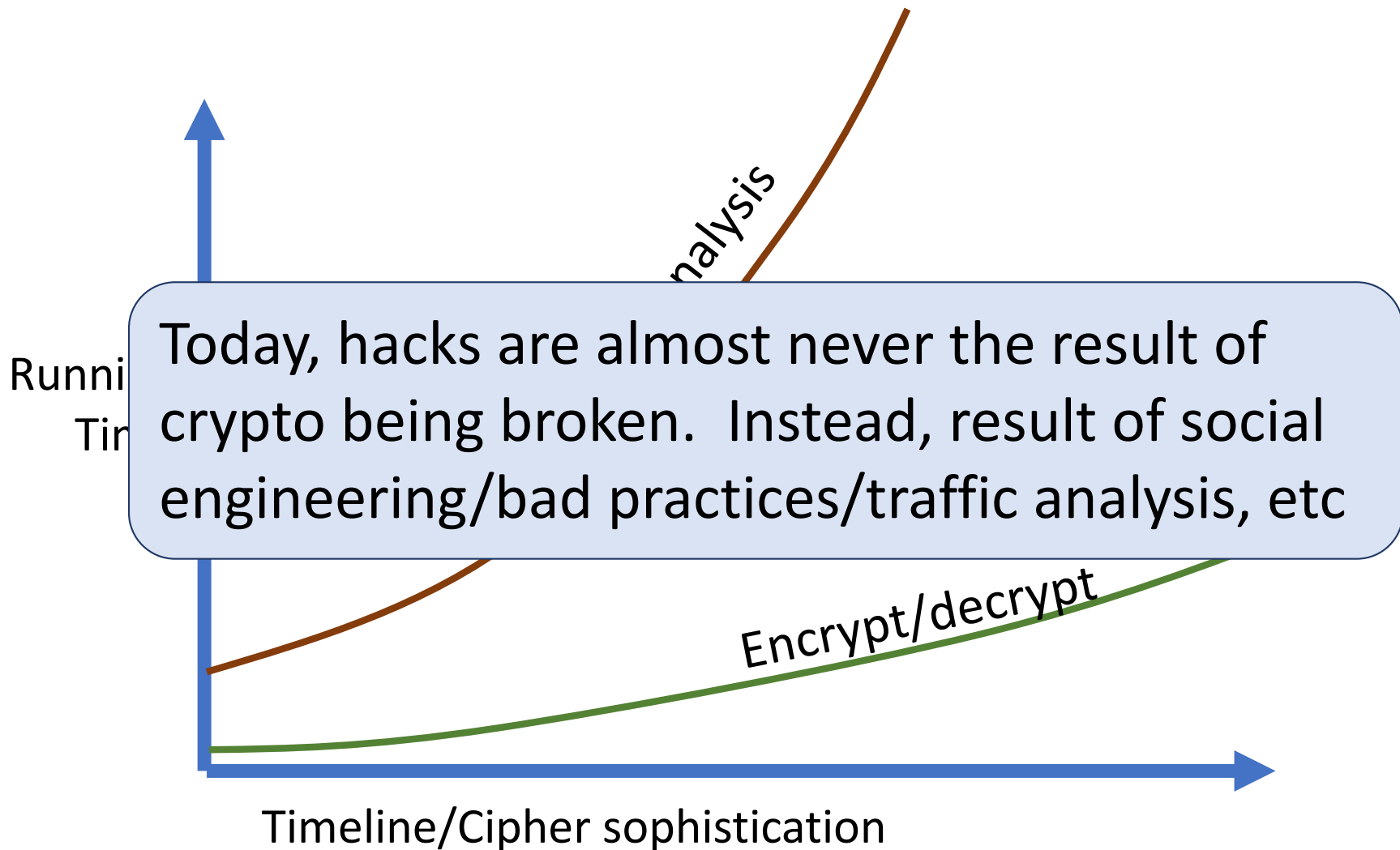
Kerckhoffs's Principle: A cryptosystem should be secure even if everything about the system, except the key, is public knowledge.

- Leaks happen. Should only have to update key, not redesign entire system
 - Even worse, cipher can potentially be reconstructed from ciphertexts
- More eyes means more likely to be secure
- Necessary for formalizing crypto

Takeaway: Importance of Computers



Takeaway: Importance of Computers



Modern Cryptography

Basics of Defining Crypto

Usually three pieces:

1. **Syntax:** what algorithms are there, what are the inputs/outputs
2. **Correctness/completeness:** how do the algorithms interact
3. **Security:** what should an adversary be permitted/prevented from doing

Formalizing Encryption (syntax and correctness)

Syntax:

- Key space \mathbf{K}
- Message space \mathbf{M}
- Ciphertext space \mathbf{C}
- **Enc: $\mathbf{K} \times \mathbf{M} \rightarrow \mathbf{C}$**
- **Dec: $\mathbf{K} \times \mathbf{C} \rightarrow \mathbf{M}$**

Correctness:

- For all $\mathbf{k} \in \mathbf{K}$, $\mathbf{m} \in \mathbf{M}$, **$\text{Dec}(\mathbf{k}, \text{Enc}(\mathbf{k}, \mathbf{m})) = \mathbf{m}$**

Example: Substitution Cipher

K? $\text{Perms}(\{a, \dots, z\})$

M? $\{a, \dots, z\}^*$

C? $\{a, \dots, z\}^*$

Enc($k, m_1 m_2 \dots$) = $k(m_1)k(m_2)\dots$

Dec($k, c_1 c_2 \dots$) = $k^{-1}(c_1)k^{-1}(c_2)\dots$

Correctness: $m_i' = k^{-1}(c_i) = k^{-1}(k(m_i)) = m_i$

Example: Transposition Cipher

K? $\text{Perms}(\{1, \dots, n\})$

M? $\{a, \dots, z\}^n$

C? $\{a, \dots, z\}^n$

Enc($k, m_1 m_2 \dots$) = $m_{k(1)} m_{k(2)} \dots$

Dec($k, c_1 c_2 \dots$) = $c_{k^{-1}(1)} c_{k^{-1}(2)} \dots$

Correctness: $m_i' = c_{k^{-1}(i)} = m_{k^{-1}(k(i))} = m_i$

Example: One-Time Pad

K? $\{0,1\}^n$

M? $\{0,1\}^n$

C? $\{0,1\}^n$

Enc(k,m) = $m \oplus k$ (more generally, **$m+k$**)

Dec(k,c) = $c \oplus k$ (more generally, **$c-k$**)

Correctness: **$m' = c \oplus k = (m \oplus k) \oplus k = m$**

Example: Vigenère Cipher

K? $\{0,1\}^\lambda$

M? $\{0,1\}^n$

C? $\{0,1\}^n$

Enc(k,m) = $(m_1 \oplus k_1) \dots (m_i \oplus k_{i \bmod \lambda}) \dots$

Dec(k,c) = $(c_1 \oplus k_1) \dots (c_i \oplus k_{i \bmod \lambda}) \dots$

Correctness:

$$m_i' = c_i \oplus k_{i \bmod \lambda} = (m_i \oplus k_{i \bmod \lambda}) \oplus k_{i \bmod \lambda} = m_i$$

Encryption Security?

Questions to think about:

What kind of messages?

What does the adversary already know?

What information are we trying to protect?

Examples:

- Messages are always either “attack at dawn” or “attack at dusk”, trying to hide which is the case
- Messages are status updates (“<person> reports <event> at <location>”). Which data is sensitive?

Encryption Security?

Questions to think about:

What kind of messages?

What does the adversary already know?

What information are we trying to protect?

Goal:

Rather than design a separate system for each use case, design a system that works in all possible settings

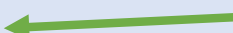
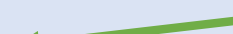
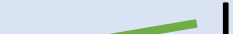
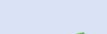
Semantic Security


Idea:

- Plaintext comes from an arbitrary distribution
- Adversary initially has some information about the plaintext
- Seeing the ciphertext should not reveal any more information
- Model unknown key by assuming it is chosen uniformly at random

(Perfect) Semantic Security

Definition: A scheme **(Enc, Dec)** is **(perfectly) semantically secure** if, for all:

- Distributions **D** on **M**  Plaintext distribution
- Functions **I: M → {0,1}***  Info adv gets
- Functions **f: M → {0,1}***  Info adv tries to learn
- Functions **A: C × {0,1}^* → {0,1}^***  Adversary

There exists a function **S: {0,1}^* → {0,1}^***  “Simulator” such that

$$\begin{aligned} \Pr[A(\text{Enc}(k,m) , I(m)) = f(m)] \\ = \Pr[S(I(m)) = f(m)] \end{aligned}$$

where probabilities are taken over $k \leftarrow K, m \leftarrow D$

Semantic Security

Captures what we want out of an encryption scheme

But, complicated, with many moving parts

Want: something simpler...

Notation

Two random variables \mathbf{X}, \mathbf{Y} over a finite set \mathbf{S} have identical distributions if, for all $\mathbf{s} \in \mathbf{S}$,

$$\Pr[\mathbf{X} = \mathbf{s}] = \Pr[\mathbf{Y} = \mathbf{s}]$$

In this case, we write

$$\mathbf{X} \stackrel{d}{=} \mathbf{Y}$$

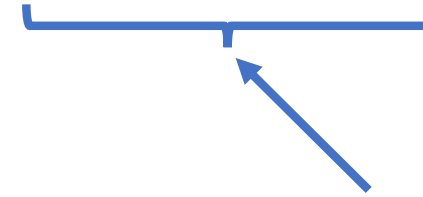
Perfect Secrecy [Shannon'49]

Definition: A scheme **(Enc, Dec)** has **perfect secrecy** if, for any two messages $\mathbf{m}_0, \mathbf{m}_1 \in \mathcal{M}$

$$\text{Enc}(\mathbf{K}, \mathbf{m}_0) \stackrel{d}{=} \text{Enc}(\mathbf{K}, \mathbf{m}_1)$$



Random variable corresponding
to uniform distribution over \mathbf{K}



Random variable corresponding
to encrypting \mathbf{m}_1 using a
uniformly random key

Semantic Security = Perfect Secrecy

Theorem: A scheme **(Enc,Dec)** is semantically secure if and only if it has perfect secrecy

Intuition

Semantic Security \Rightarrow Perfect Secrecy

- Side information: message $\in \{m_0, m_1\}$
- Adversary trying to learn which one

Perfect Secrecy \Rightarrow Semantic Security

- $S(I(m)) = A(\text{Enc}(k,0), I(m))$

Perfect vs. Semantic Security

Semantic security is the “right” notion to intuitively capture the desired security goals

Perfect is much simpler and easier to reason about

Fortunately, we know both are identical

⇒ perfect security is almost always what is used

Any perfectly/semantically secure schemes?

Perfect Security of One-Time Pad

Fix any message $\mathbf{m} \in \{0,1\}^n$, ciphertext $\mathbf{c} \in \{0,1\}^n$

$$\begin{aligned}\Pr_k[\text{Enc}(k, \mathbf{m}) = \mathbf{c}] &= \Pr_k[\mathbf{k} \oplus \mathbf{m} = \mathbf{c}] \\ &= \Pr_k[\mathbf{k} = \mathbf{m} \oplus \mathbf{c}] \\ &= 2^{-n}\end{aligned}$$

Therefore, for any \mathbf{m} , $\text{Enc}(\mathbf{K}, \mathbf{m}) = \text{uniform dist.}$

In particular, for any $\mathbf{m}_0, \mathbf{m}_1$,

$$\text{Enc}(\mathbf{K}, \mathbf{m}_0) \stackrel{d}{=} \text{Enc}(\mathbf{K}, \mathbf{m}_1)$$

Insecurity of Substitution/Transposition

$m_0 = aa$

$m_1 = ab$

$\Pr[\text{Enc}(K, m_0) \text{ has 2 identical characters}] = 1$

$\Pr[\text{Enc}(K, m_1) \text{ has 2 identical characters}] = 0$

Proper Use Case for Perfect Security

- Message can come from any distribution ✓
- Adversary can know anything about message ✓
- Encryption hides anything ✓
- But, definition only says something about an adversary that sees a single message ✗
 - ⇒ If two messages, no security guarantee
- Assumes no side-channels ✗
- Assumes key is uniformly random ✗

Next Time

Limitations of perfect secrecy/one-time pad:

- Key length
- Multiple-message security

Reminders

- Fill out OH poll
- HW1, PR1 to be released soon

Find teams for projects (up to 4)