Homework 1

1 Problem 1

Consider the following **Experiment 1**. Sample a random bit b, and produce the state $|\phi\rangle = |b\rangle$.

Prove that the output of this experiment cannot be described by a pure state alone. In particular, suppose toward contradiction that there was a pure state $|\psi\rangle$ that described the output above experiment. Show that in fact $|\psi\rangle$ can be distinguished from $|\phi\rangle$. To do so, devise a unitary matrix U (based on $|\psi\rangle$) such that if you apply U to $|\psi\rangle$ or $|\phi\rangle$ and measure, the outcomes of the measurements will have different probability distributions.

A state $|\phi\rangle$ sampled from a probability distribution like the procedure above is known as a *mixed* state.

2 Problem 2

Consider the following **Experiment 2**. Sample a random bit b, and produce the state $|\phi'\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{(-1)^b}{\sqrt{2}}|1\rangle$. Equivalently, sample $|\phi\rangle$ according to **Experiment 1**, and then apply the Hadamard transformation.

Show that there is no unitary transformation followed by a measurement that distinguishes $|\phi\rangle$ sampled from **Experiment 1** from $|\phi'\rangle$ sampled from **Experiment 2**. That is, show that for any unitary U, if you apply U to $|\phi\rangle$ and measure, or apply U to $|\phi'\rangle$ and measure, the probability distributions in the two cases are identical.

3 Problem 3

The right way to describe a mixed state is by a *density matrix*. Suppose that state $|\phi_i\rangle$ is sampled with probability p_i. Then the density matrix is the matrix

$$\rho = \sum_i p_i |\phi_i\rangle\langle\phi_i|$$

(Remember that the notation $\langle\phi_i|$ means the row vector that is the conjugate transpose of $|\phi_i\rangle$)

It turns out that the density matrix captures all statistical information about the mixed state. That is, no sequence of unitary operations and measurements can distinguish two mixed states with the same density matrix, and for any two states with different density matrices, there is a unitary and measurement that distinguish the two (with some non-zero probability).

(a) A pure state is a special case of a mixed state where the probability distribution has support on only a single state. Therefore, pure states also have density matrices. What is special about the density matrix for a pure state?

(b) What is the density matrix for the output of **Experiment 1**? Combined with part (a), Why does this show that the state can be distinguished from any pure state?

(c) What is the density matrix for the output of **Experiment 2**?

(d) Given an arbitrary mixed state, suppose you apply a unitary U to the state. Explain how to transform the corresponding density matrix?

(e) Given an arbitrary mixed state, suppose you measure the state. Let q_j be the probability the measurement gives j. What is q_j in terms of the density matrix for the state? Hint: start by analyzing pure states, and then build up to a mixed state from there.

(f) Mixed states are useful for characterizing the state that remains after performing a measurement. What is the density matrix for the state that results from measuring a pure state $|\phi\rangle$, in terms of the entries in $|\phi\rangle$?

(g) The result of measuring a mixed state is another mixed state. How does measuring transform the density matrix?

(h) Suppose I have a mixed state over a 2-qubit system. Now I measure the first qubit. How does this affect the density matrix?

(i) Consider a mixed qubit state, defined by an arbitrary distribution over potentially many pure states $|\phi_i\rangle$. Prove that this mixed state is equivalent to a mixed state whose probability distribution is over just two pure states.

4 Problem 4

Here, we will discuss how to generalize our notion of a measurement. Consider a quantum state over set B of size n. Fix an arbitrary orthonormal basis $C = \{|b_1\rangle, \ldots, |b_n\rangle\}$ for the space $\mathbb{C}^n$. That is, $|b_i\rangle$ are all orthogonal vectors.

The result of measuring $|\phi\rangle$ in basis C is the following. First, the measurement will output i with probability $|\langle b_i|\phi\rangle|^2$. Then, the state will collapse to $|b_i\rangle$.

The definition of measurement we saw in class is the special case where C is the computational basis.

(a) Explain why the probability distribution over i is in fact a probability distribution (that is, the probabilities sum to 1).

(b) Show that measuring in basis C is equivalent to (1) applying a unitary U, (2) measuring in the computational basis, and (3) applying a unitary U'. Thus, without loss of generality, we can usually just consider measuring in the computational basis.

5 Problem 5

Even more general measurements are possible. Here, we will consider what are known as *projective* measurements. Such a measurement is specified by a set of *projection* matrices $P_1, \ldots, P_k$. A projection matrix P is a Hermitian matrix (meaning $P^\dagger = P$) such that $P^2 = P$. We will additionally need that $\sum_i P_i$ is the identity matrix.

The result of applying the projective measurement to $|\phi\rangle$ is the following. First, the measurement will output i with probability $\langle\phi|P_i|\phi\rangle$. Then, the state will collapse to $\frac{P_i|\phi\rangle}{\sqrt{\langle\phi|P_i|\phi\rangle}}$.

(a) Explain why the probability distribution over i is in fact a probability distribution (that is, the probabilities are non-negative and sum to 1).

(b) Show that, if a projective measurement is applied twice to the same state, the outcomes of the measurement will be the same both times.

(c) Show that a measurement in basis C can be described as a projective measurement.

(d) Consider a 2-qubit state, and consider measuring the first qubit. Describe this partial measurement at a projective measurement.

(e) Let ρ be the density matrix for a mixed state, and ρ' the density matrix resulting from applying the projective measurement. What is ρ' in terms of ρ?