COS433/Math 473: Cryptography

Mark Zhandry

Princeton University

Spring 2017

Previously

Pseudorandom Functions and Permutaitons

Modes of Operation

Pseudorandom Functions

Functions that "look like" random functions

Syntax:

- Key space $\{0,1\}^\lambda$
- Domain X (usually $\{0,1\}^m$, m may depend on λ)
- Co-domain/range Y (usually $\{0,1\}^n$, may depend on λ)
- Function $F:\{0,1\}^\lambda \times X \rightarrow Y$

Pseudorandom Permutations

(also known as block ciphers)

Functions that "look like" random **permutations**

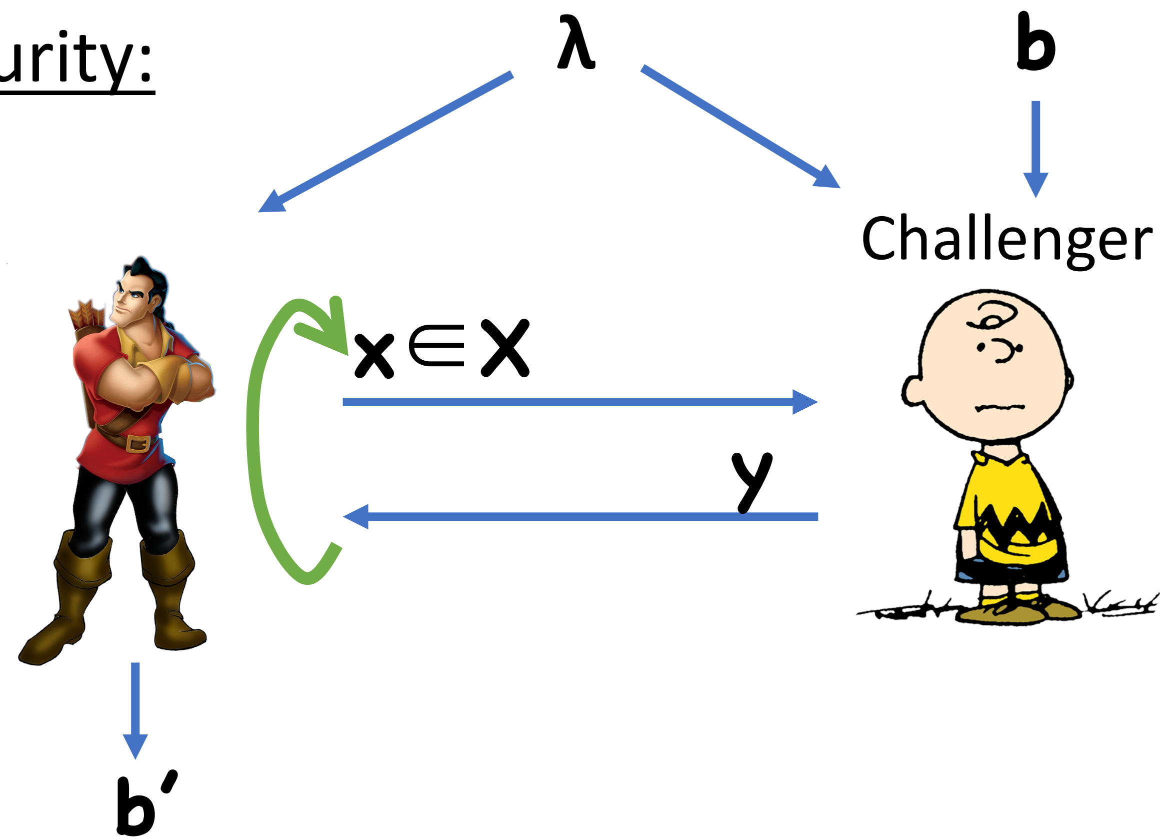
Syntax:

- Key space $\{0,1\}^\lambda$
- Domain X (usually $\{0,1\}^n$, n usually depends on λ)
- Range X
- Function $F:\{0,1\}^\lambda \times X \to X$
- Function $F^{-1}:\{0,1\}^\lambda \times X \to X$

Correctness: $\forall$ $k,x,\ F^{-1}(k,\ F(k,\ x)\ ) = x$

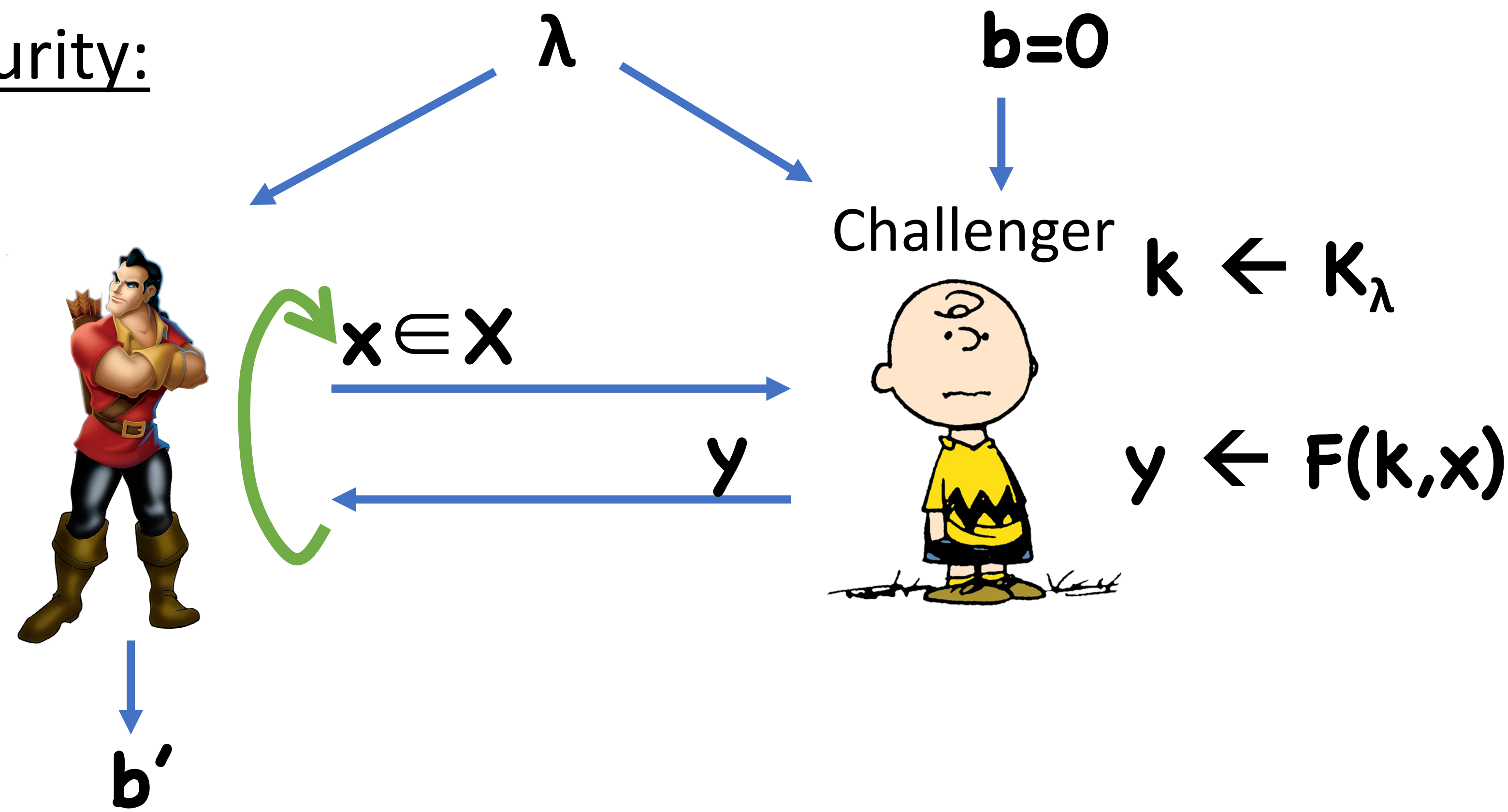
Pseudorandom Permutations

Security:



Pseudorandom Permutations

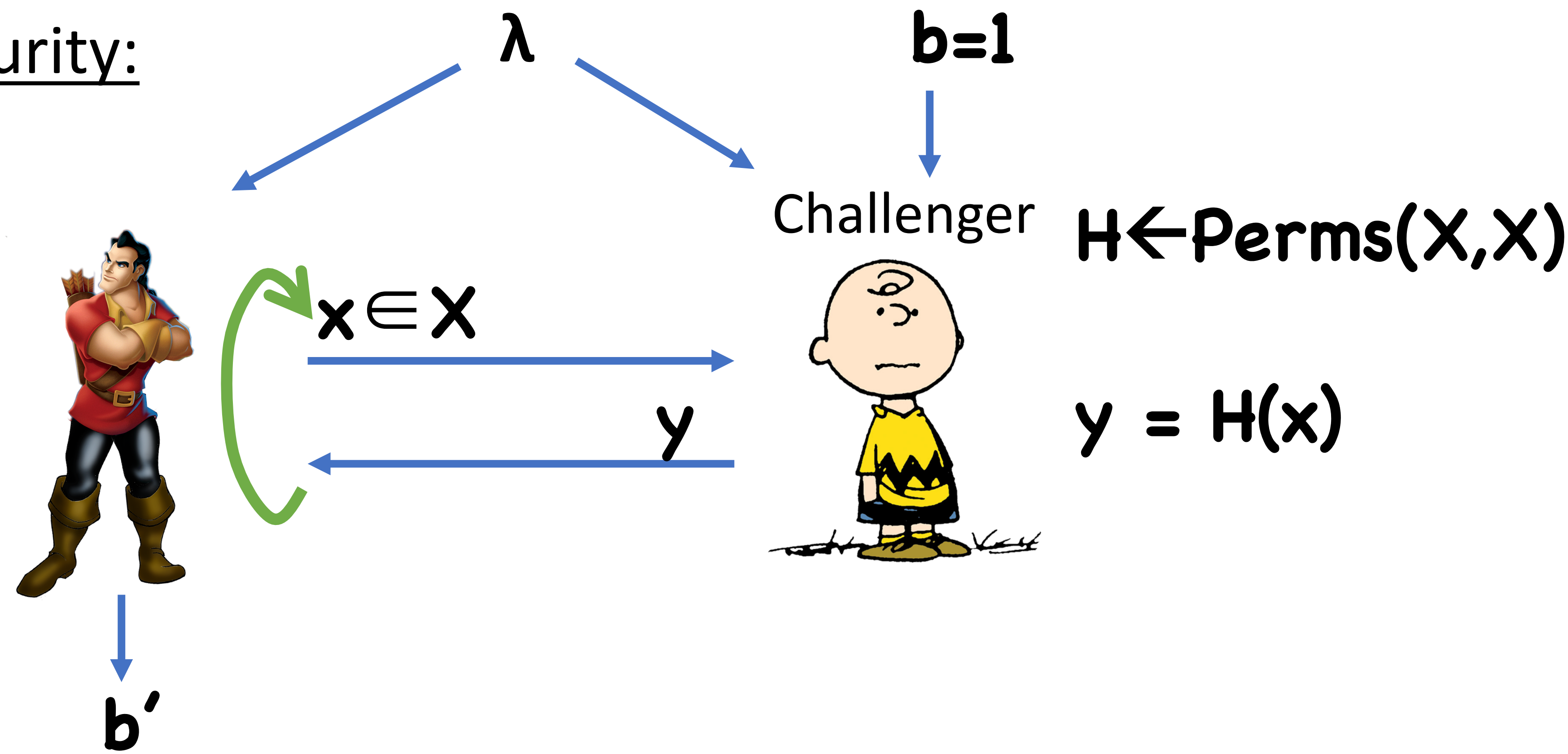
Security:



PRF-Exp$_0$( **, λ)**

Pseudorandom Permutations

<u>Security:</u>

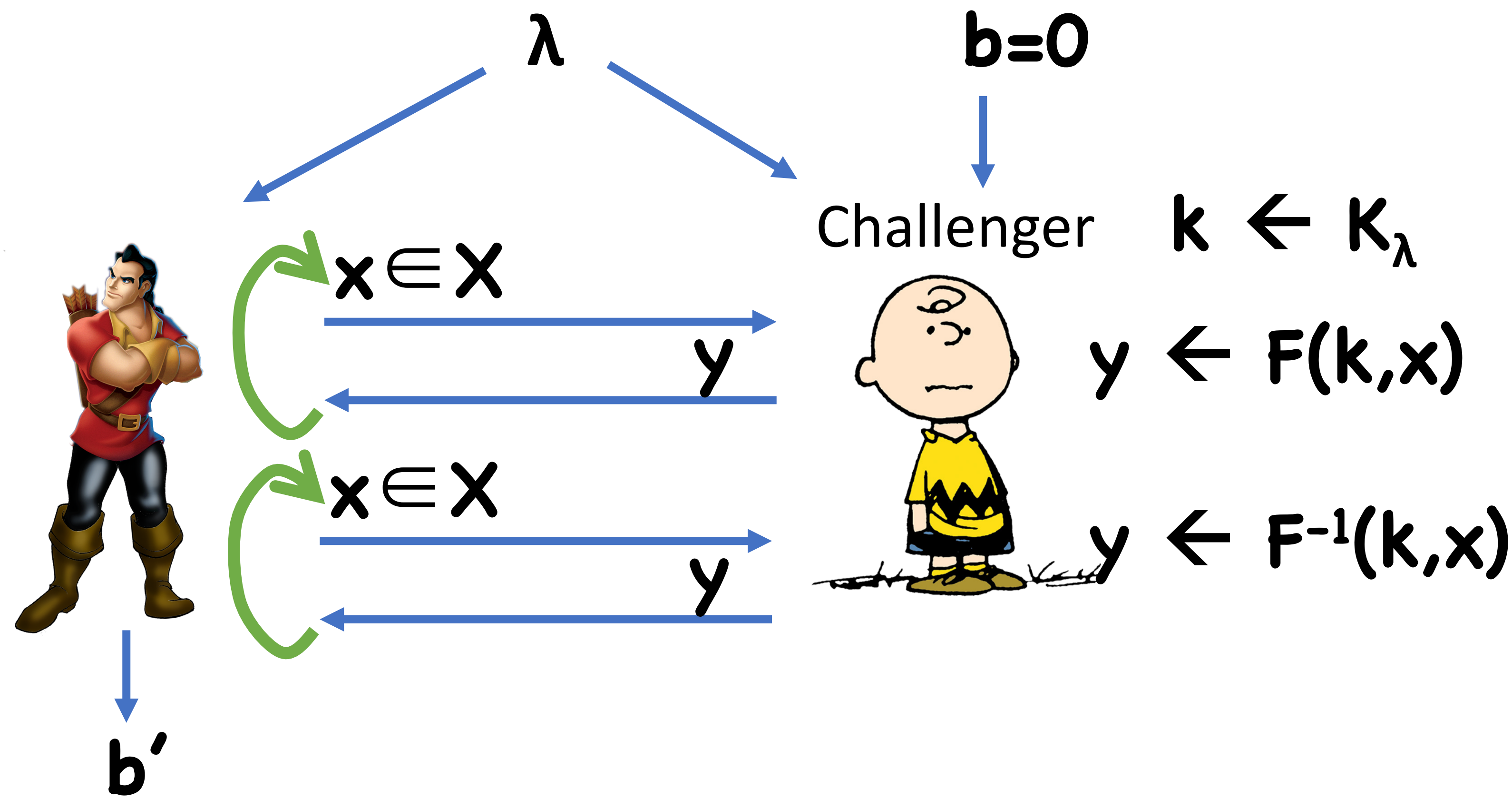


PRF-Exp$_1$**(**  **, λ)**

Theorem: A PRP (F, F^{-1}) is secure iff F is a secure as a PRF

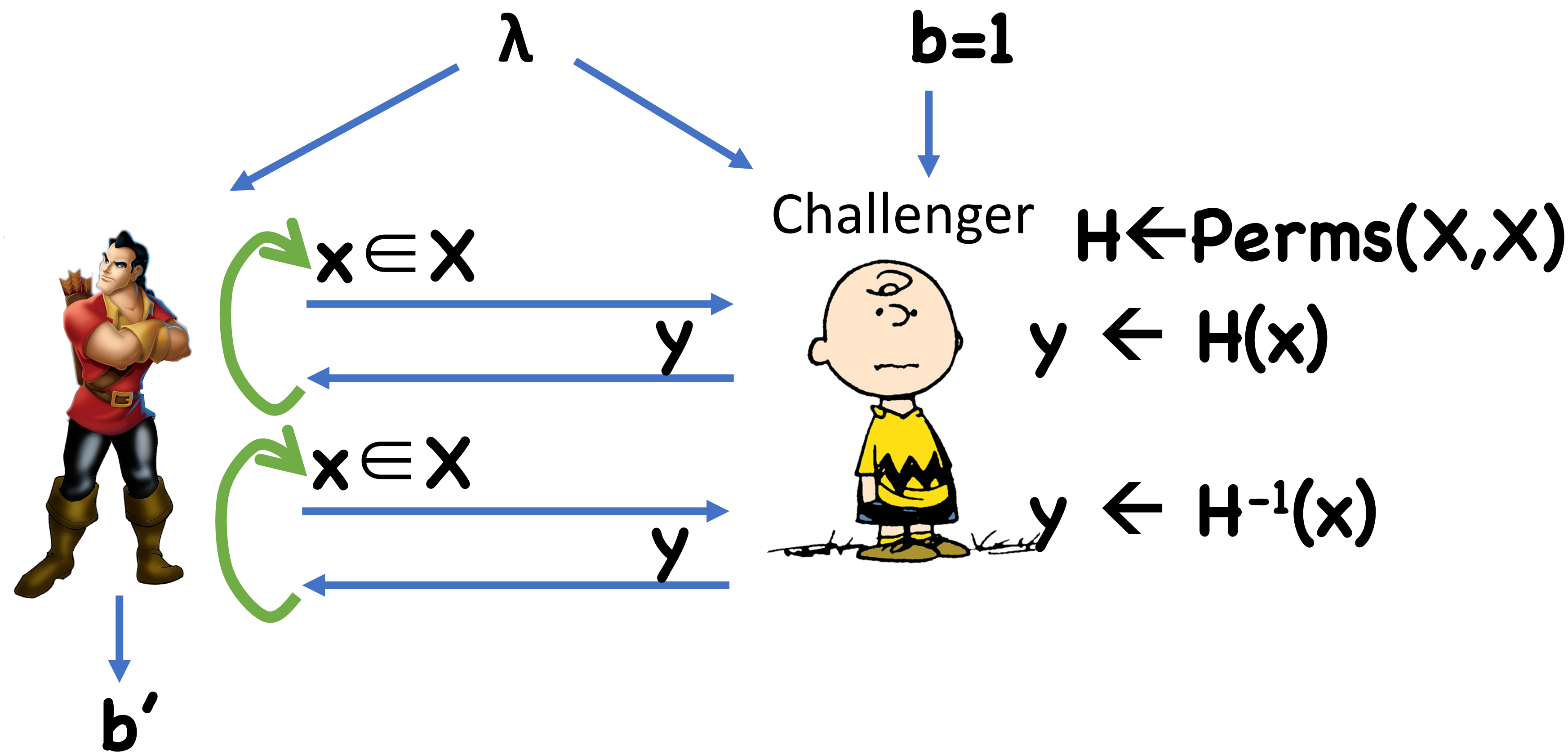
Theorem: There are secure PRPs (F, F^{-1}) where (F^{-1}, F) is insecure

Strong PRPs



PRF-Exp₀(, λ)

Strong PRPs



PRF-Exp$_1$( **, λ)**

Theorem: If (F, F^{-1}) is a strong PRP, then so is (F^{-1}, F)

PRPs vs PRFs

In practice, PRPs are the central building block of most crypto

- Also PRFs
- Can build PRGs
- Very versatile

Today

Constructing PRPs

Today, we are going to ignore negligible, and focus on concrete parameters

- E.g. 128 bit blocks
- Adversary running time $<<2^{128}$
- Etc.

Difficulties

$2^n!$ Permutations on **n**-bit blocks

$\Rightarrow$ **$\approx n2^n$** bits to write down random perm.

Reasonable for very small **n** (e.g. **$n<20$**), but totally infeasible for large **n** (e.g. **$n=128$**)

Challenge:

- Design permutations with small description that "behave like" random permutations

Difficulties

For a random permutation **H**, **H(x)** and **H(x′)** are (essentially) independent random strings

- Even if **x** and **x′** differ by just a single bit

Therefore, for a random key **k**, changing a single bit of **x** should "affect" all output bits of **F(k,x)**

Definition: For a function $\mathbf{H:\{0,1\}^n \rightarrow \{0,1\}^n}$, we say that bit **i** of the input affects bit **j** of the output if:

For a random $\mathbf{x_1,\ldots,x_{i-1},x_{i+1},\ \ldots,\ x_n}$, if we let
$\mathbf{y=H(x_1\ldots x_{i-1}0x_{i+1}\ldots x_n)}$ and
$\mathbf{z=H(x_1\ldots x_{i-1}1x_{i+1}\ldots x_n)}$
Then $\mathbf{y_j \neq z_j}$ with probability $\mathbf{\approx 1/2}$

Theorem: If (F, F^{-1}) is a secure PRP, then with (with "high" probability over the key k), for the function $F(k, \bullet)$, every bit of input affects every bit of output

Proof:

- For random permutations this is true
- If bit i did not affect bit j, we can construct an adversary that distinguishes F from random

Confusion/Diffusion Paradigm

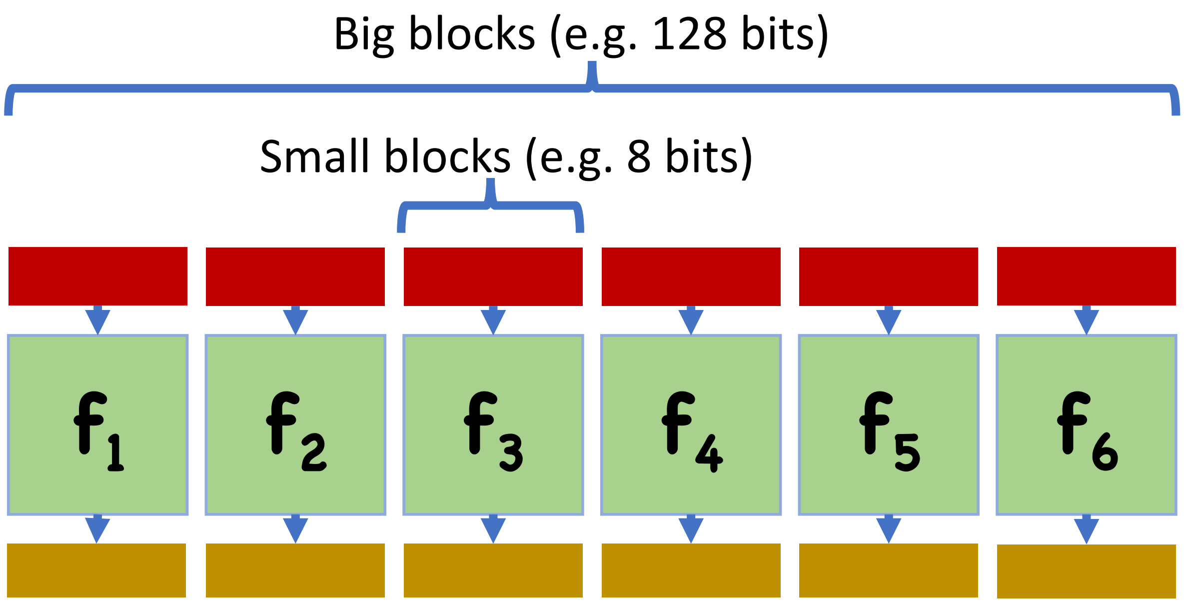
Confusion/Diffusion Paradigm

Goal: build permutation for large blocks from permutations for small blocks

- Small block perms can be made truly random
- Hopefully result is pseudorandom

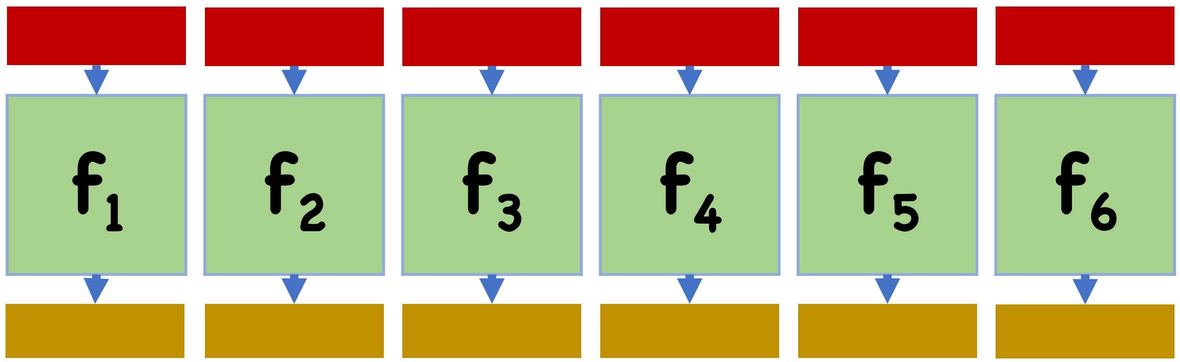
Confusion/Diffusion Paradigm

First attempt: break blocks into smaller blocks, apply smaller permutation blockwise



Key: description of f_1, f_2,…

Confusion/Diffusion Paradigm

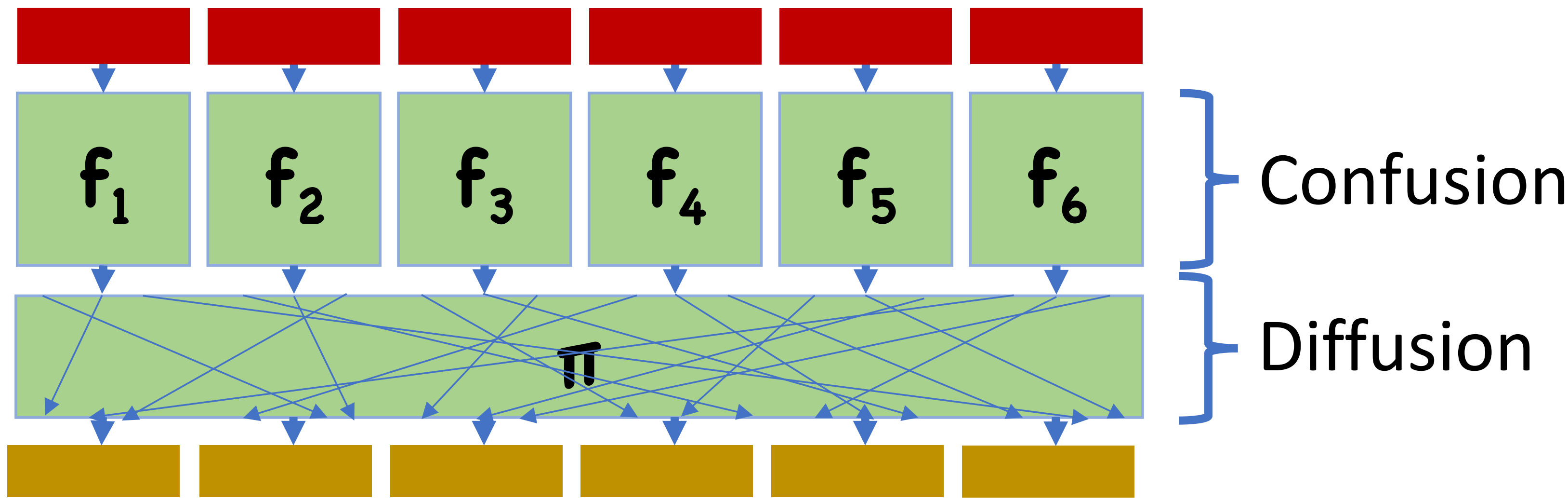


Is this a secure PRP?

- Key size: $\approx (8 \times 2^8) \times (128/8) = 2^{15}$, so reasonable
- Running time: a few table lookups, so efficient
- Security?

Confusion/Diffusion Paradigm

Second attempt: shuffle output bits



Is this a secure PRP?

- Key size: $\approx 2^{15} + 128 \times \text{Log } 128 \approx 2^{15}$
- Running time: a few table lookups
- Security?

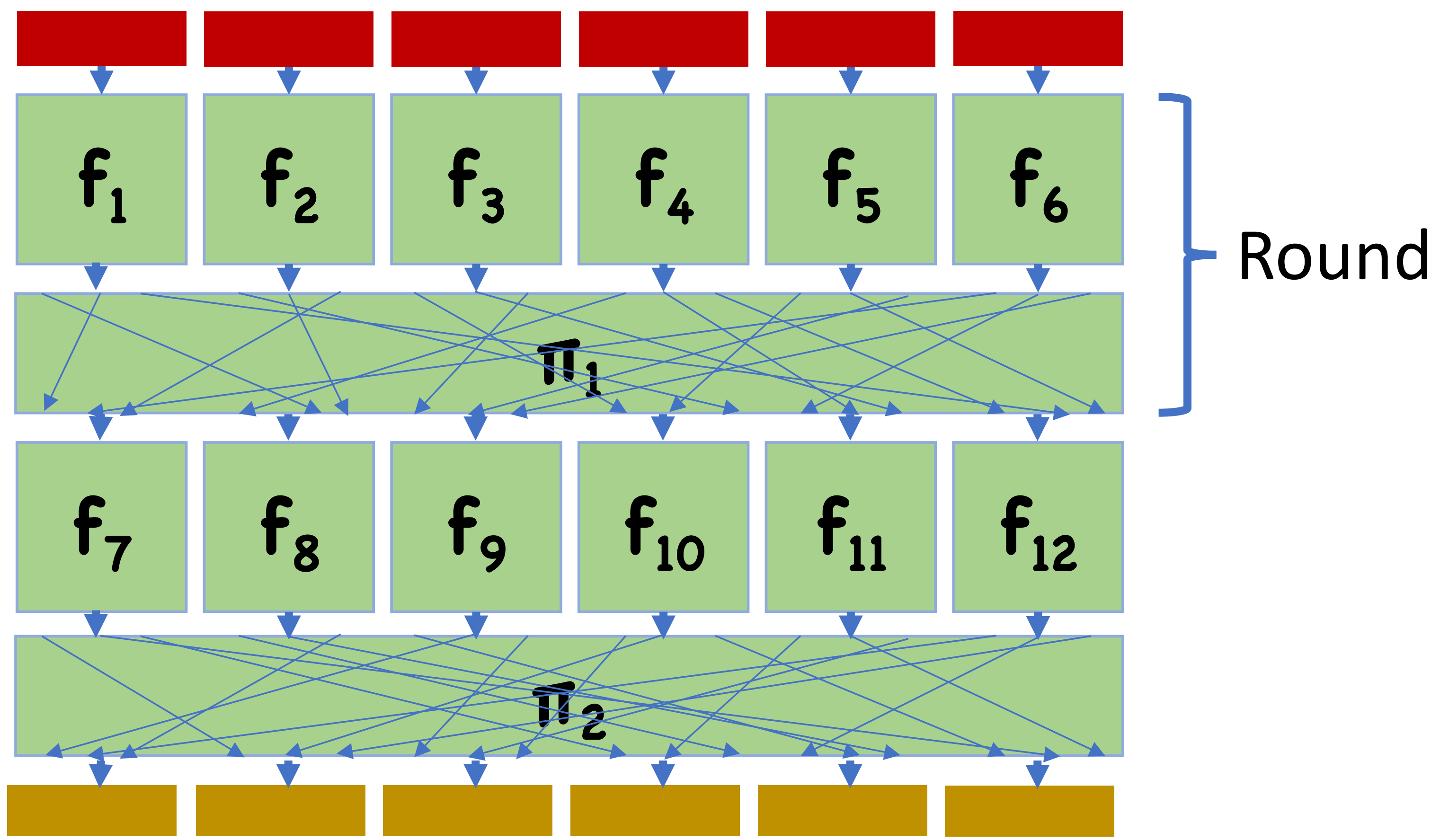
Confusion/Diffusion Paradigm

While confusion/diffusion is not secure, we’ve made progress

- Each bit affects 8 output bits

Next step: repeat!

Confusion/Diffusion Paradigm



Confusion/Diffusion Paradigm

With 2 rounds,
- Each bit affects 64 output bits

With 3 rounds, all 128 bits are affected

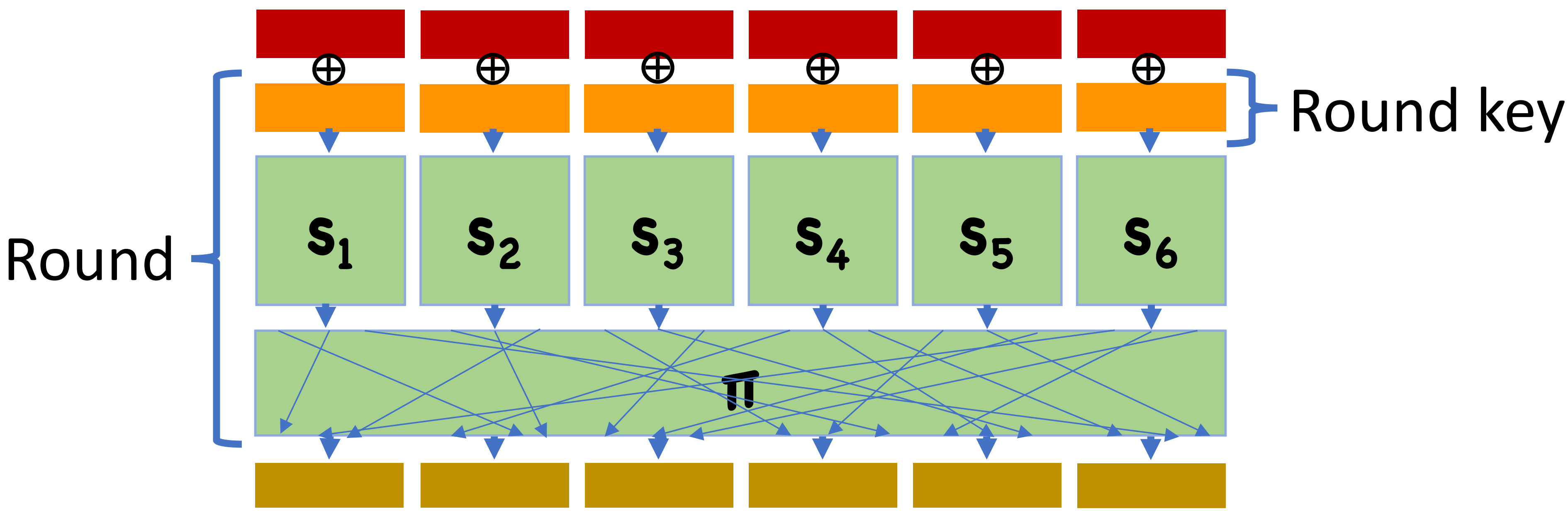
Repeat a few more times for good measure
- Why is 3 rounds still not enough?

Substitution Permutation Networks

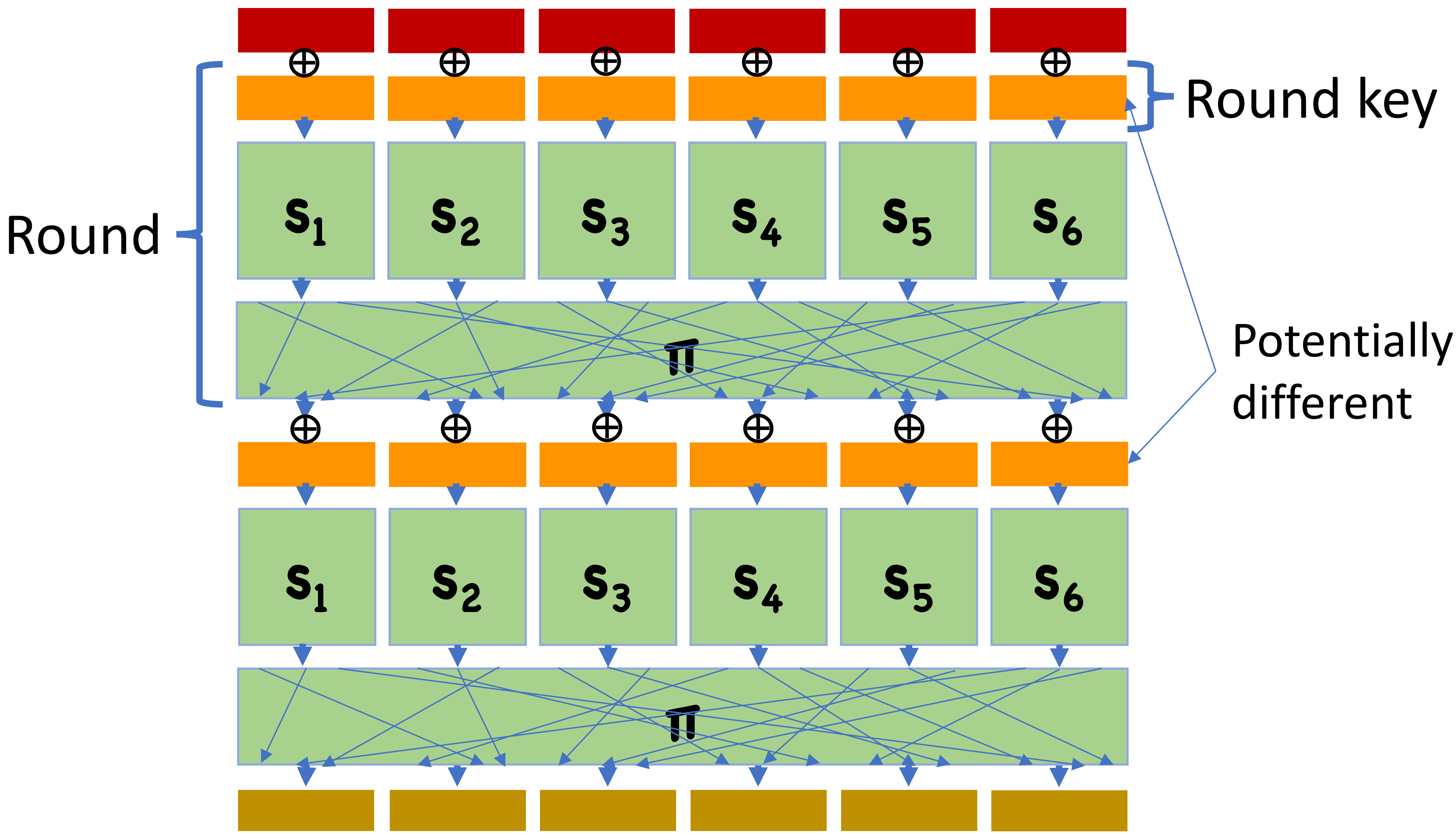
Variant of previous construction

- Fixed public permutations for confusion (called a substitution box, or S-box)
- Fixed public permutation for diffusion (called a permutation box, or P-box)
- XOR "round key" at beginning of each round

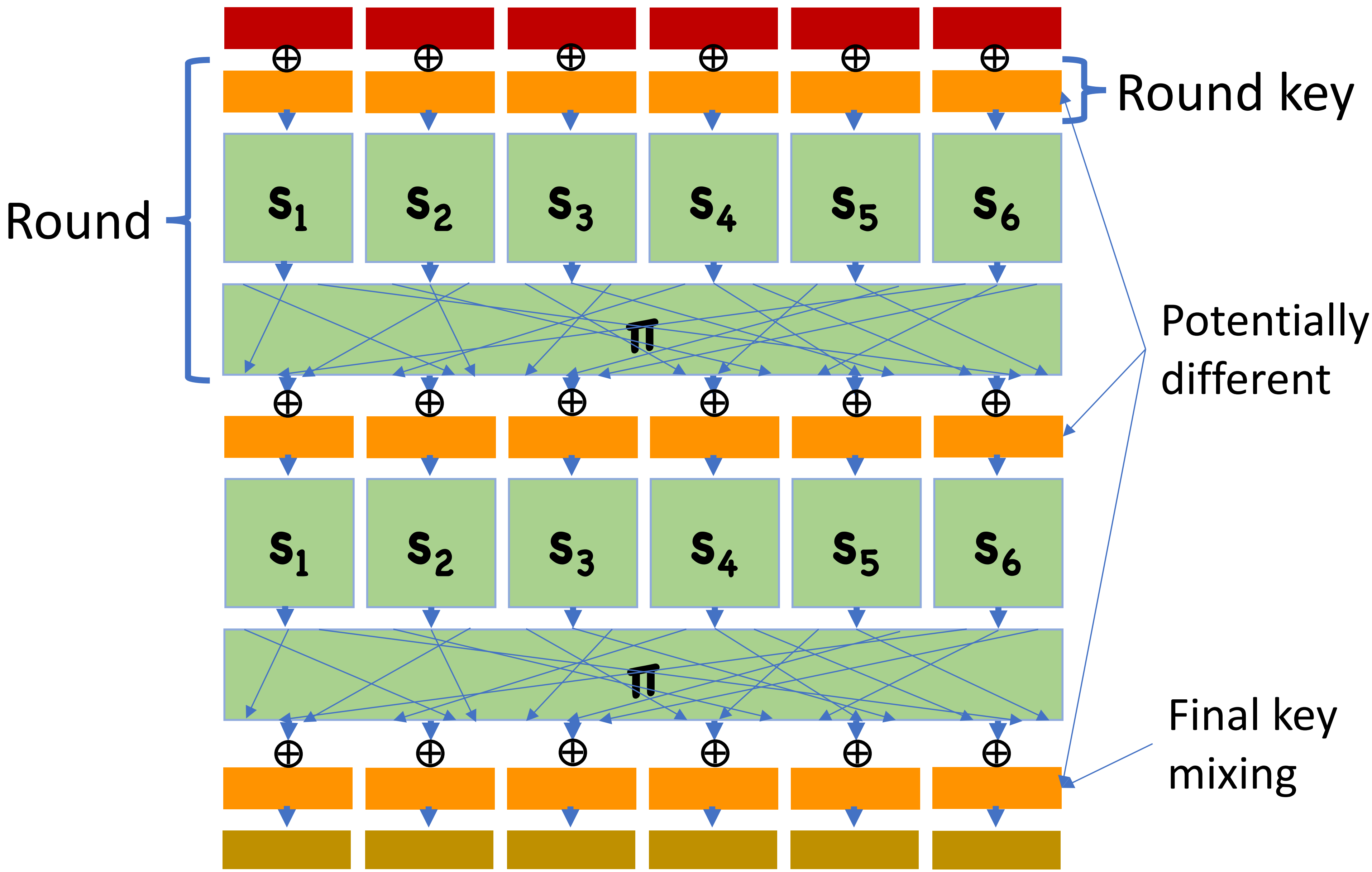
Substitution Permutation Networks



Substitution Permutation Networks



Substitution Permutation Networks



Substitution Permutation Networks

To specify a network, must:

- Specify S-boxes
- Specify P-box
- Specify key schedule (how round keys are derived from master)

Choice of parameters can greatly affect security

Designing SPNs

Avalanche Affect:

- Need S-boxes and mixing permutations to cause every input bit to ”affect” every output bit

One way to guarantee this:

- Changing any bit of S-box input causes at least 2 bits of output to change
- Mixing permutations send outputs of S-boxes into at least 2 different S-boxes for next round
- Sufficiently many rounds are used
- At least how many rounds should be used?

Designing SPNs

For strong PRPs, need avalanche in reverse too

- Changing one bit of output of S box changes at least 2 bits of input
- Mixing permutations take inputs for next round from at least two different S-box outputs

Designing S-Boxes

Random?

- Let x,x' be two distinct 4-bit values
- $\Pr[S(x)$ and $S(x')$ differ on a single bit$] = 4/15$
- Very high probability that some pair of inputs will have outputs that differ on a single bit

Therefore, must carefully design S-boxes rather than choose at random

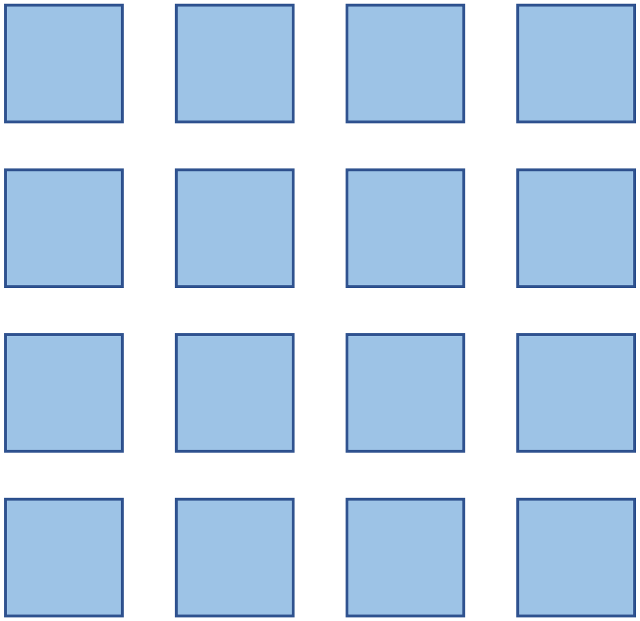
Linearity?

Can S-Boxes be linear?

- That is, $\mathbf{S(x_0)} \oplus \mathbf{S(x_1)} = \mathbf{S(x_0 \oplus x_1)}$?

AES

State = **4×4** grid of bytes



AES

One fixed S-box, applied to each byte

- Step 1: multiplicative inverse over finite field $\mathbb{F}_8$
- Step 2: fixed affine transformation
- Implemented as a simple lookup table

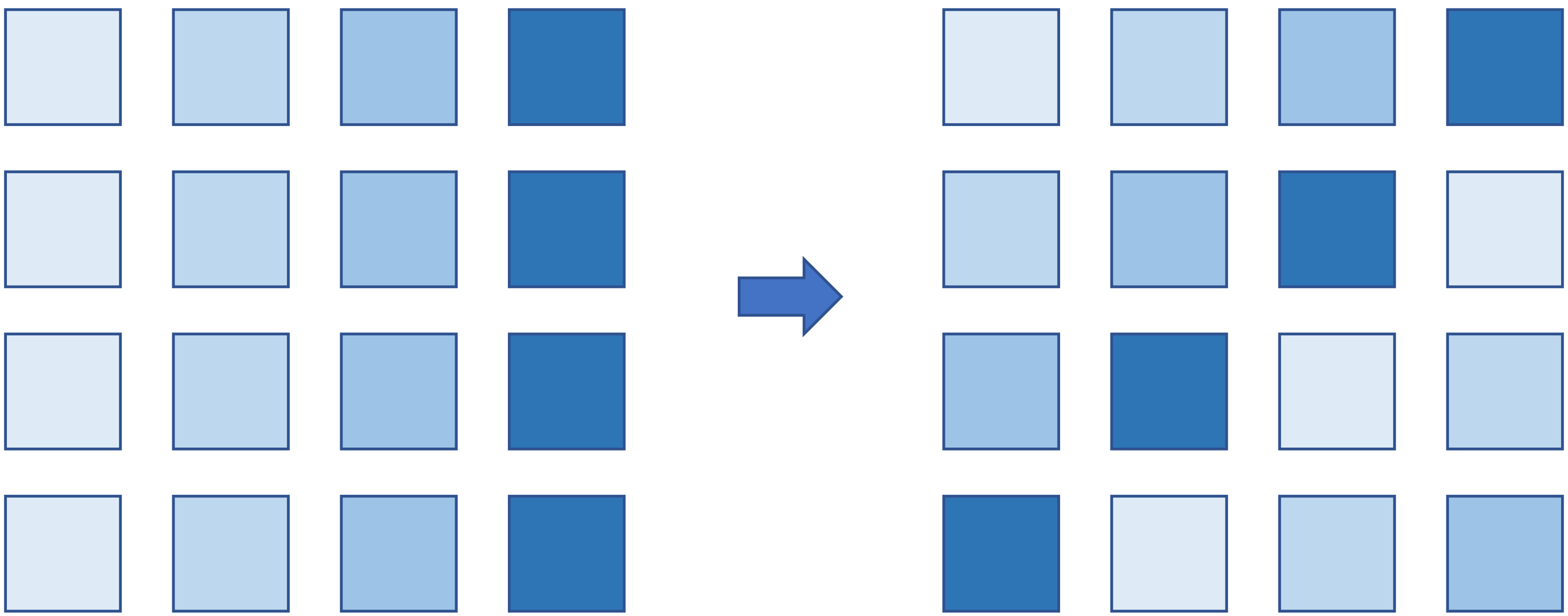
AES

Diffusion (not exactly a P-box):

- Step 1: shift rows
- Step 2: mix columns

AES

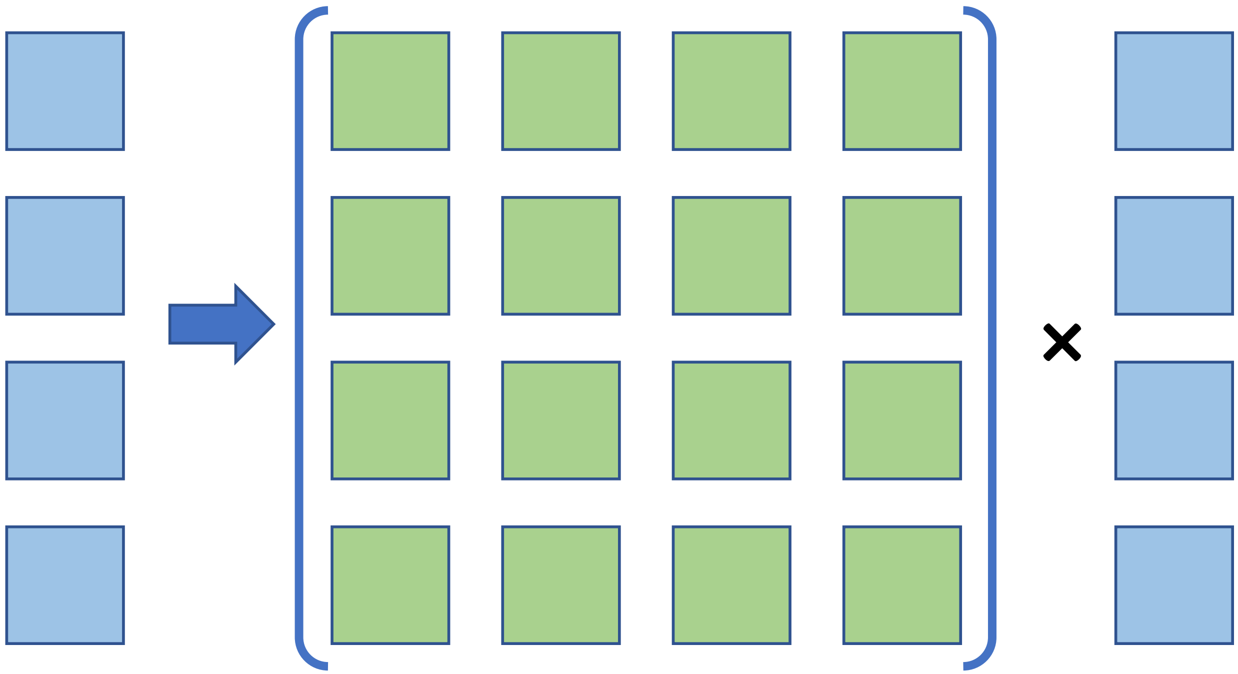
Shift Rows:



AES

Mix Columns

- Each byte interpreted as element of $\mathbb{F}_8$
- Each column is then a length-4 vector
- Apply fixed linear transformation to each column



AES

Number of rounds depends on key size

- 128-bit keys: 10 rounds
- 192-bit keys: 12 rounds
- 256-bit keys: 14 rounds

Key schedule:

- Won’t describe here, but involves more shifting, S-boxes, etc
- Can think of key schedule as a weak PRG

Fiestel Networks

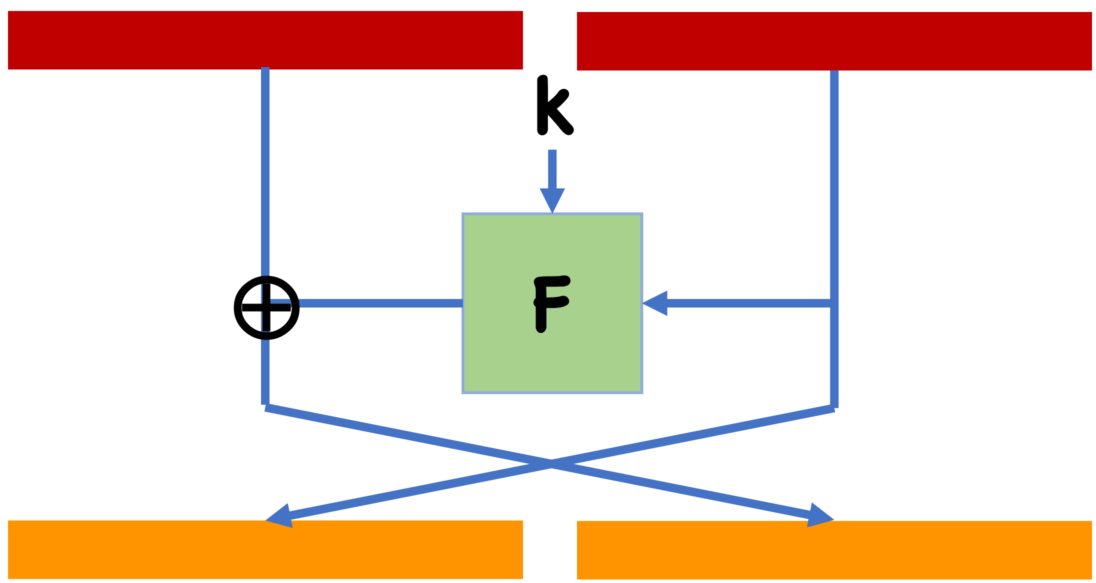
Feistel Networks

Designing permutations with good security properties is hard

What if instead we could built a good permutation from a function with good security properties…

Feistel Network

Convert functions into permutations



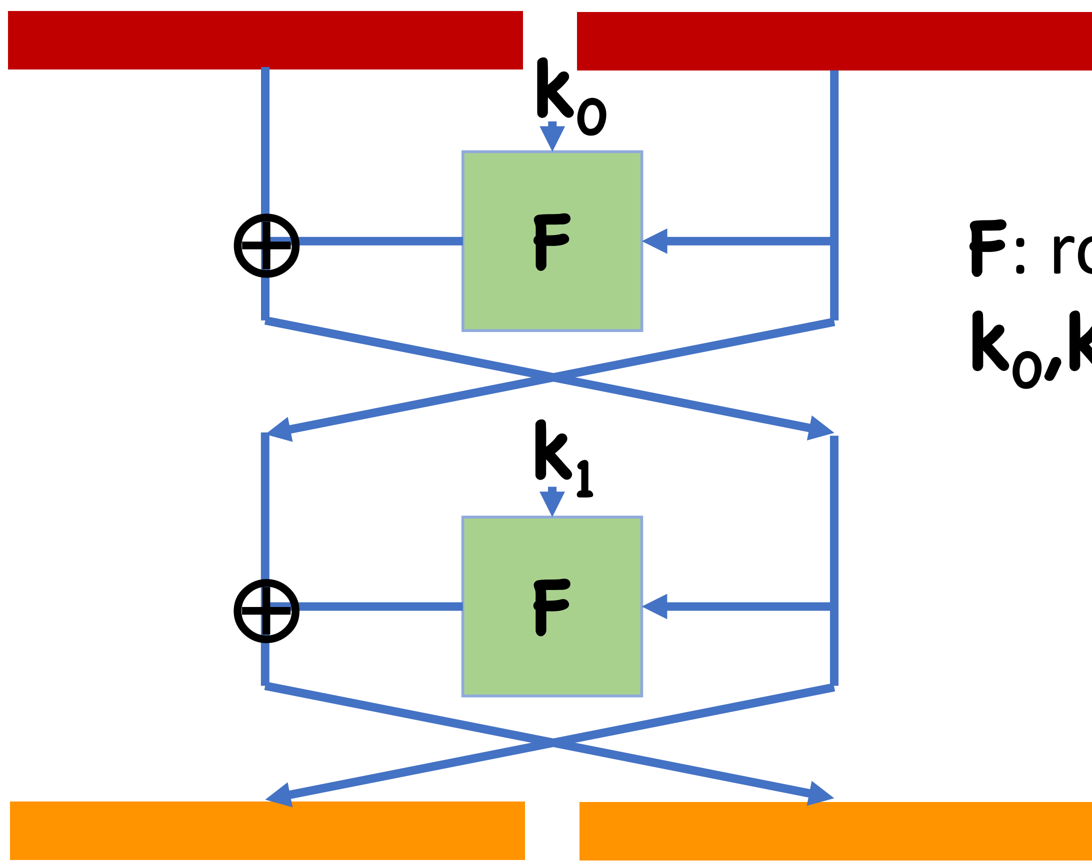
F public

k secret

Can this possibly give a secure PRP?

Feistel Network

Convert functions into permutations



F: round function

k_0,k_1: round keys

Feistel Network

Depending on specifics of round function, different number of rounds may be necessary

- Number of rounds must always be at least 3
- (Need at least 4 for a strong PRP)
- Maybe need even more for weaker round functions

Luby-Rackoff

3- or 4-round Feistel where round function is a PRF

Theorem: If F is a secure PRF, then 3 rounds of Feistel (with independent round keys) give secure PRP. 4 rounds give a strong PRP

- Proof non-trivial, won’t be covered in this class

Constructing Round Functions

Ideally, ”random looking” functions

Similar ideas to constructing PRPs

- Confusion/diffusion
- SPNs, S-boxes, etc

Key advantage is that we no longer need the functions to be permutations

- S-boxes can be non-permutations

DES

Block size: 64 bits

Key size: 56 bits

Rounds: 16

!!!

DES

Key Schedule:

- Round keys are just 48-bit subsets of master key

Round function:

- Essentially an SPN network

DES S-Boxes

8 different S-boxes, each

- 6-bit input, 4-bit output
- Table lookup: 2 bits specify row, 4 specify column

- Each row contains every possible 4-bit output
- Changing one bit of input changes at least 2 bits of output

DES History

Designed in the 1970's

- At IBM, with the help of the NSA
- At the time, many in academia were suspicious of NSA's involvement
  - Mysterious S-boxes
  - Short key length
- Turns out, S-box probably designed well
  - Resistant to "differential cryptanalysis"
  - Known to IBM and NSA in 1970's, but kept secret
- Essentially only weakness is the short key length
  - Maybe secure in the 1970's, definitely not today

DES Security Today

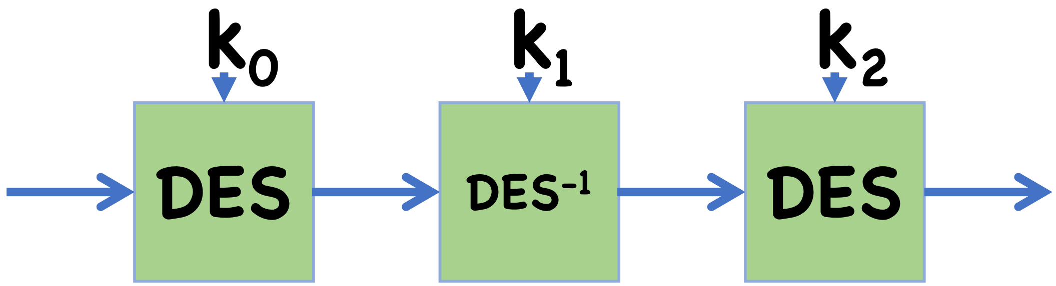
Seems like a good cipher, except for its key length and block size

What's wrong with a small block size?

- Remember for e.g. CTR mode, IV is one block
- If two identical IV's seen, attack possible
- After seeing q ciphertext, probability of repeat IV is roughly $q^2/2^{\text{block length}}$
- Attack after seeing ≈ billion messages

3DES: Increasing Key Length

3DES key = Apply DES three times with different keys



Why three times?

- Next time: "meet in the middle attack" renders 2DES no more secure than 3DES

Why inverted second permutation?

Limitations of Feistel Networks

Turns out Feistel requires block size to be large

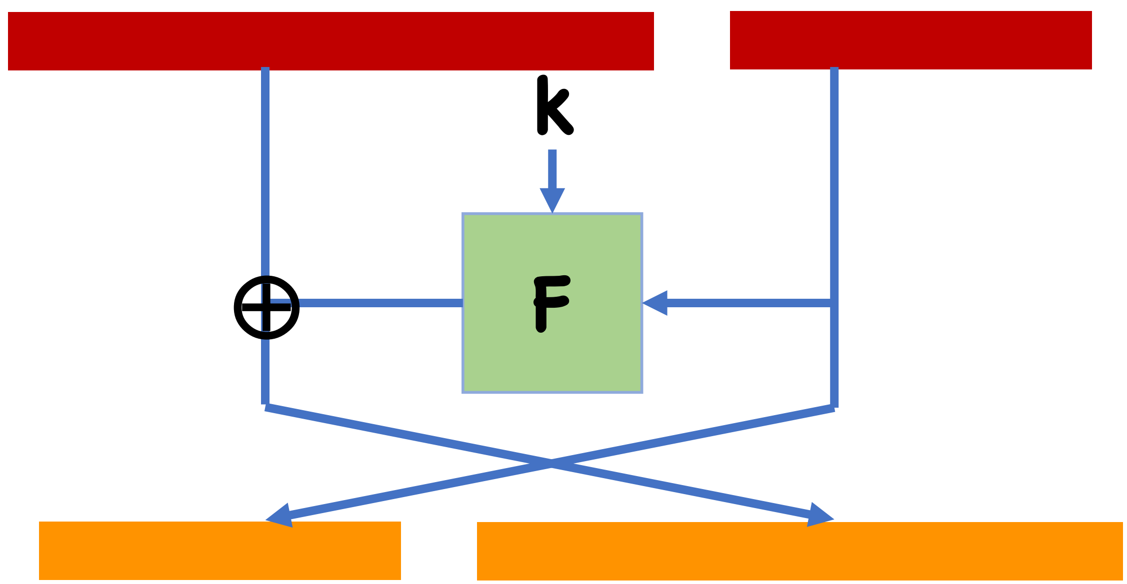
- If number of queries ~$2^{\text{block size}/2}$, can attack

Format preserving encryption:

- Encrypted data has same form as original
- E.g. encrypted SSN is an SSN
- Useful for encrypting legacy databases

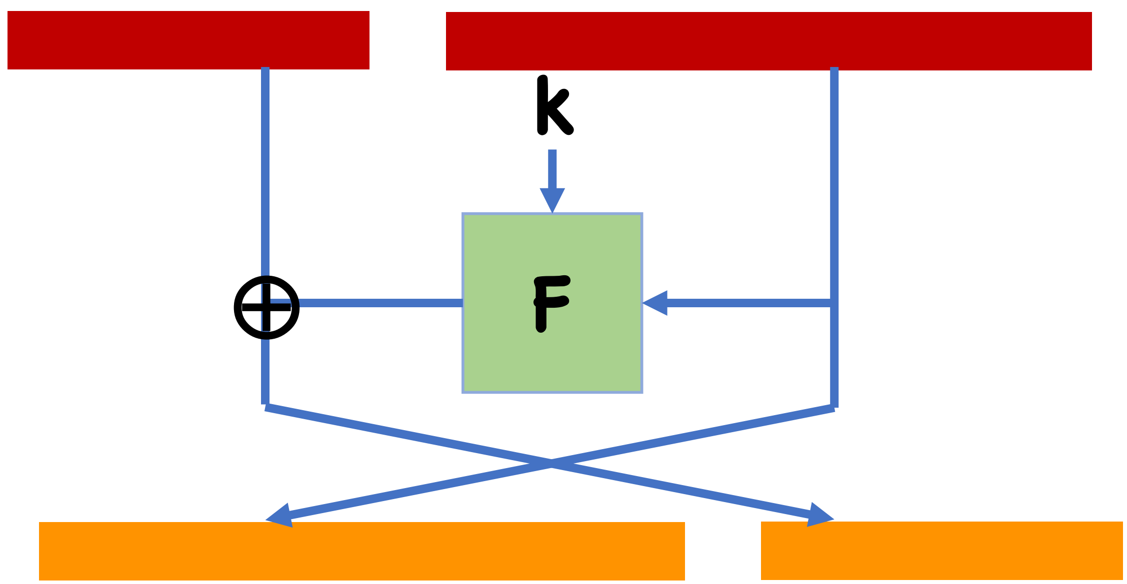
Sometimes, want a very small block size

Unbalanced Feistel



“Target heavy”

Unbalanced Feistel



“Source heavy”

Unbalanced Feistel

Taken to the extreme (where source or target is just 1 bit), one these is insecure, regardless of the round function

Which one?

Next Time

Attacks on Block Ciphers