COS433/Math 473: Cryptography

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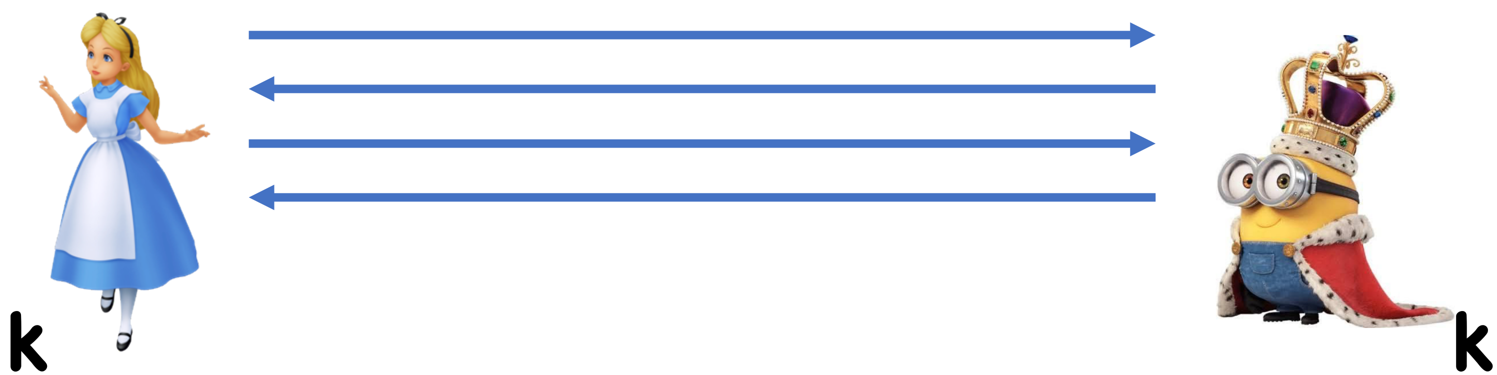
Previously

Exchanging keys and public key encryption

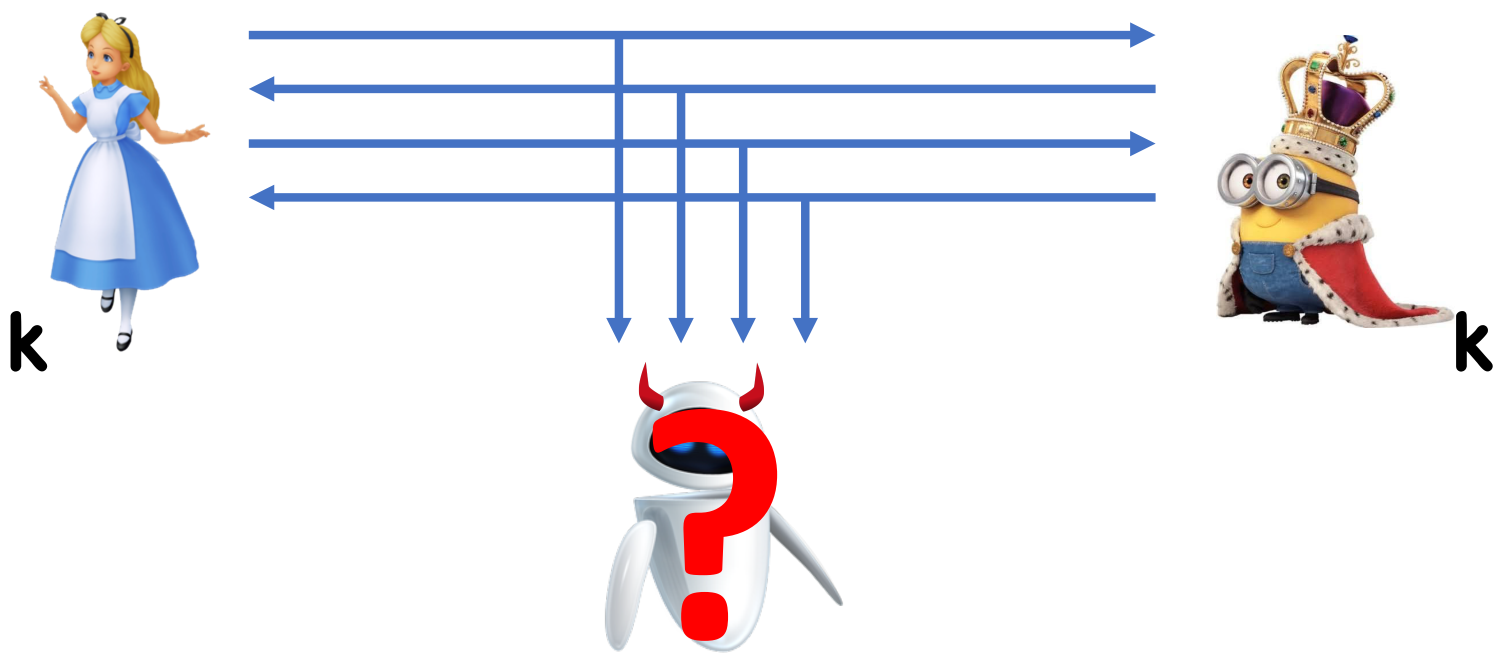
Public Key Distribution

Public Key Distribution

Public Key Distribution

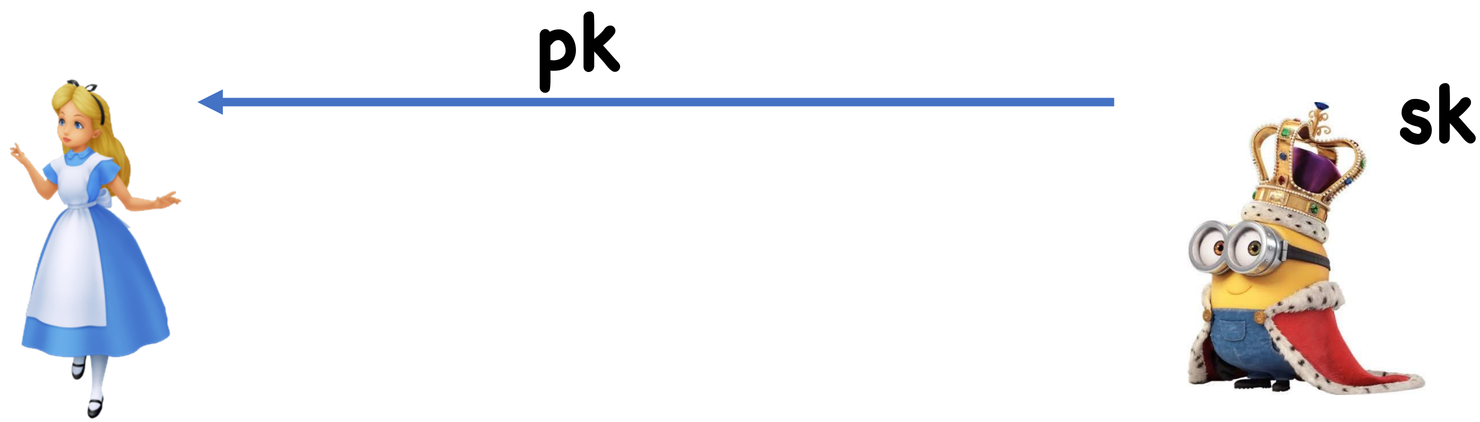


Public Key Distribution



Public Key Encryption

Public Key Encryption



Public Key Encryption

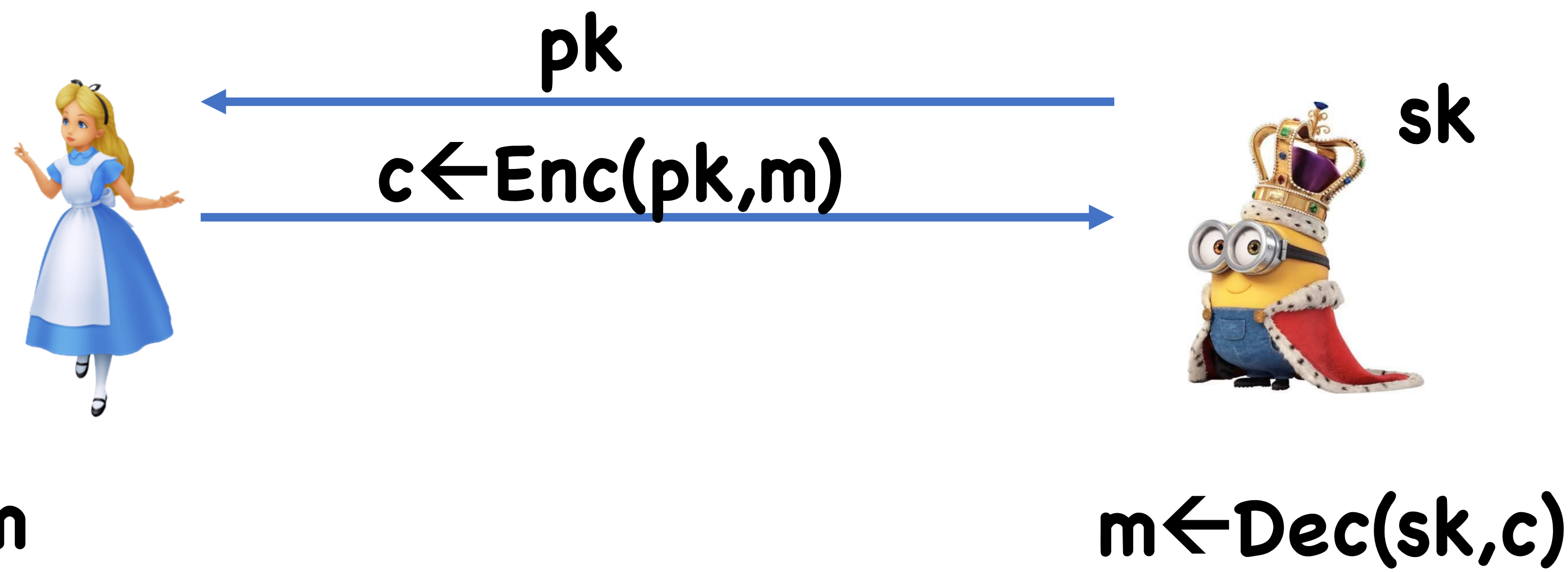
pk

$c \leftarrow Enc(pk, m)$

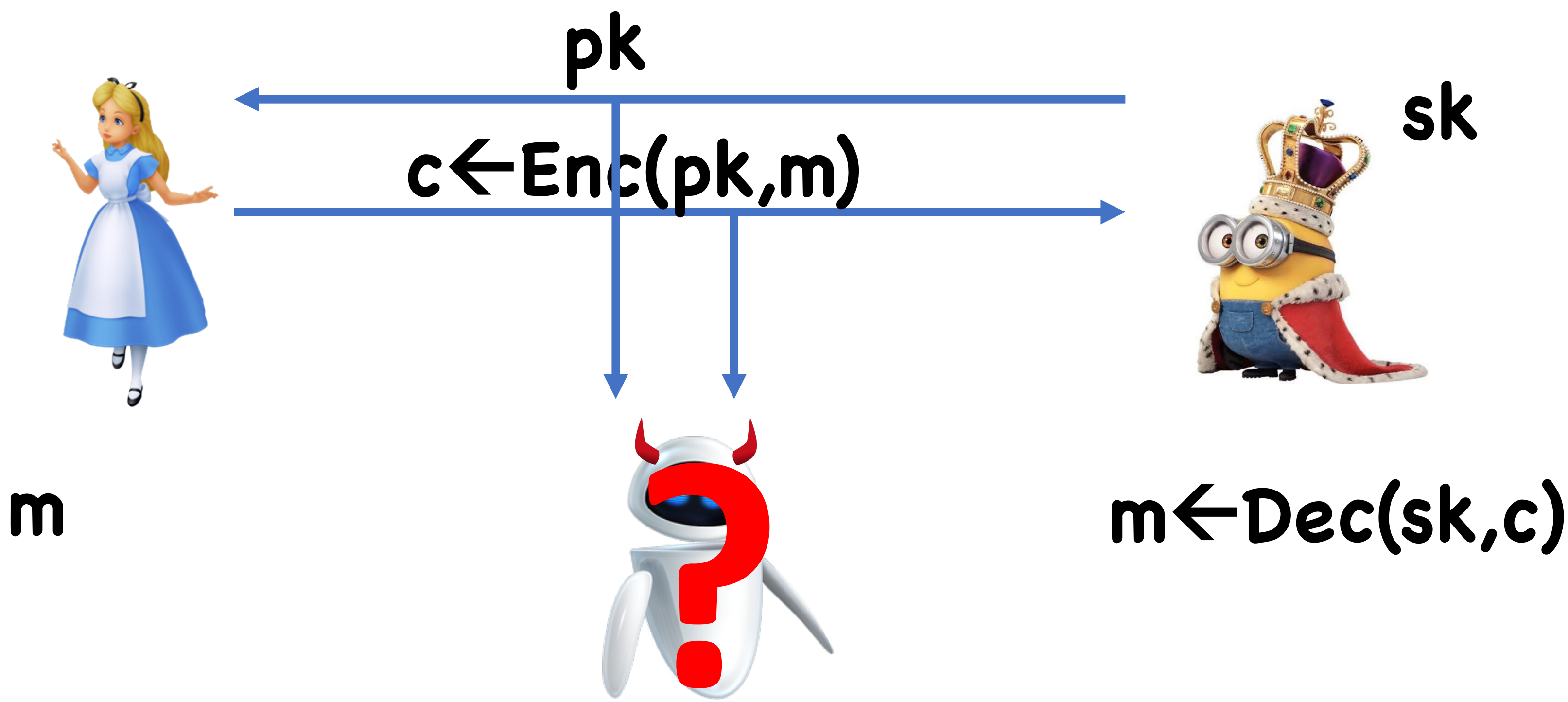
sk

m

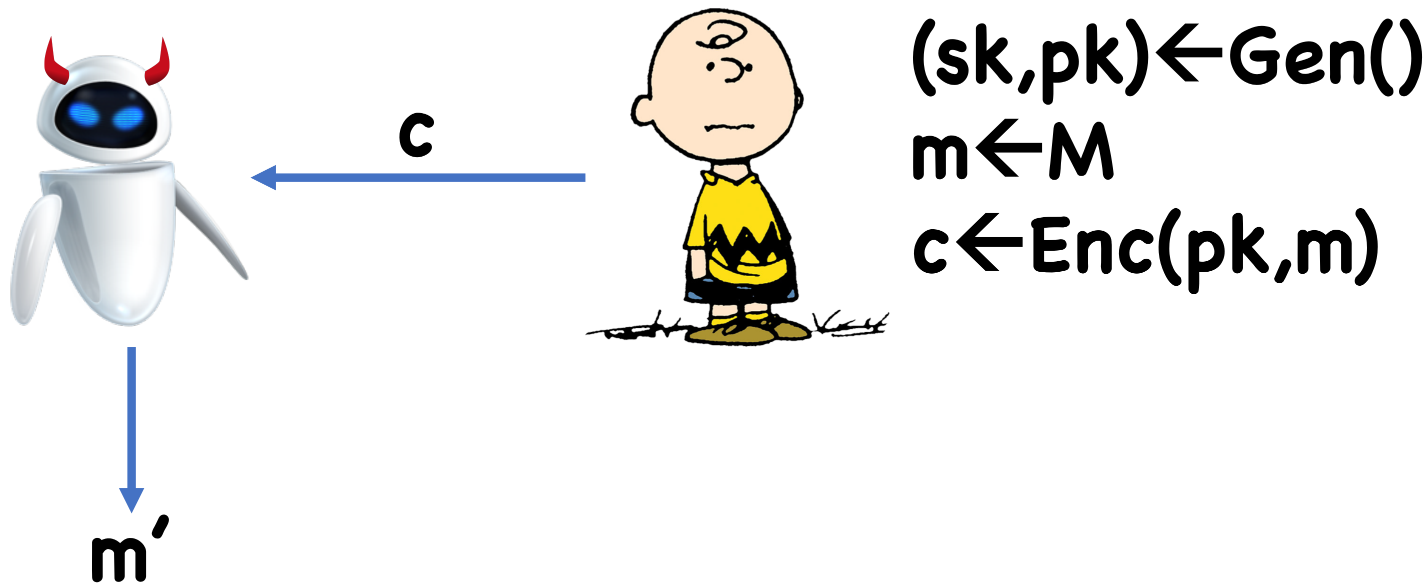
Public Key Encryption



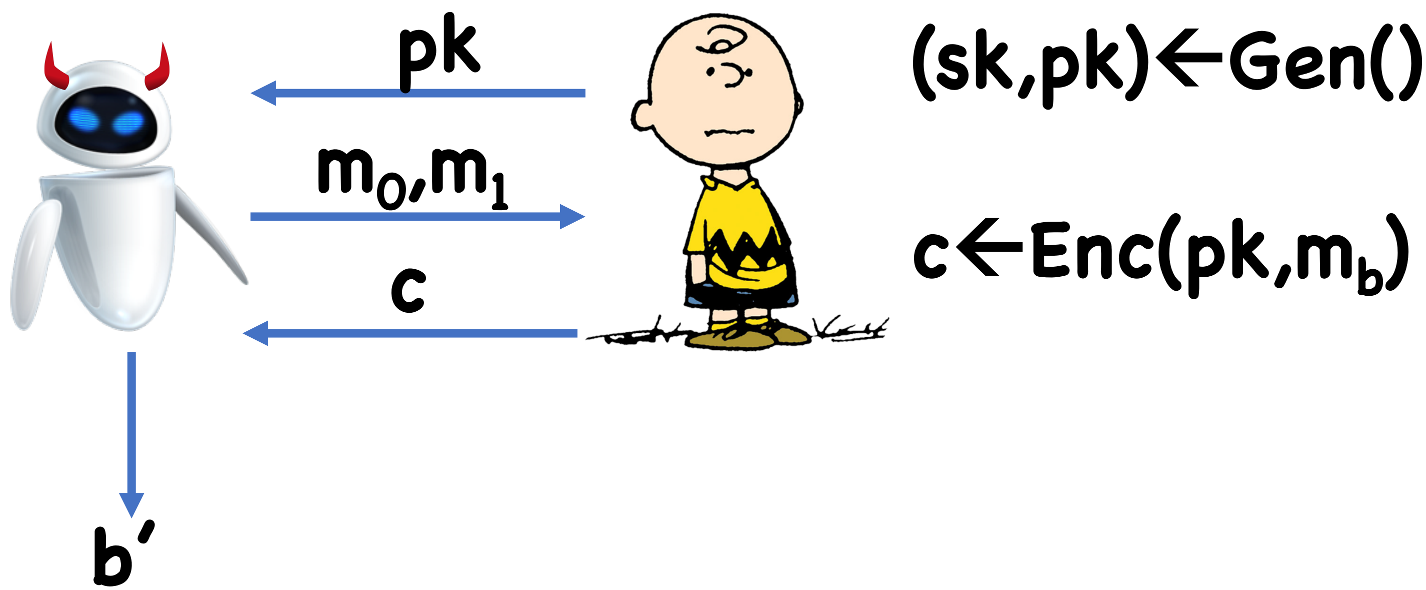
Public Key Encryption



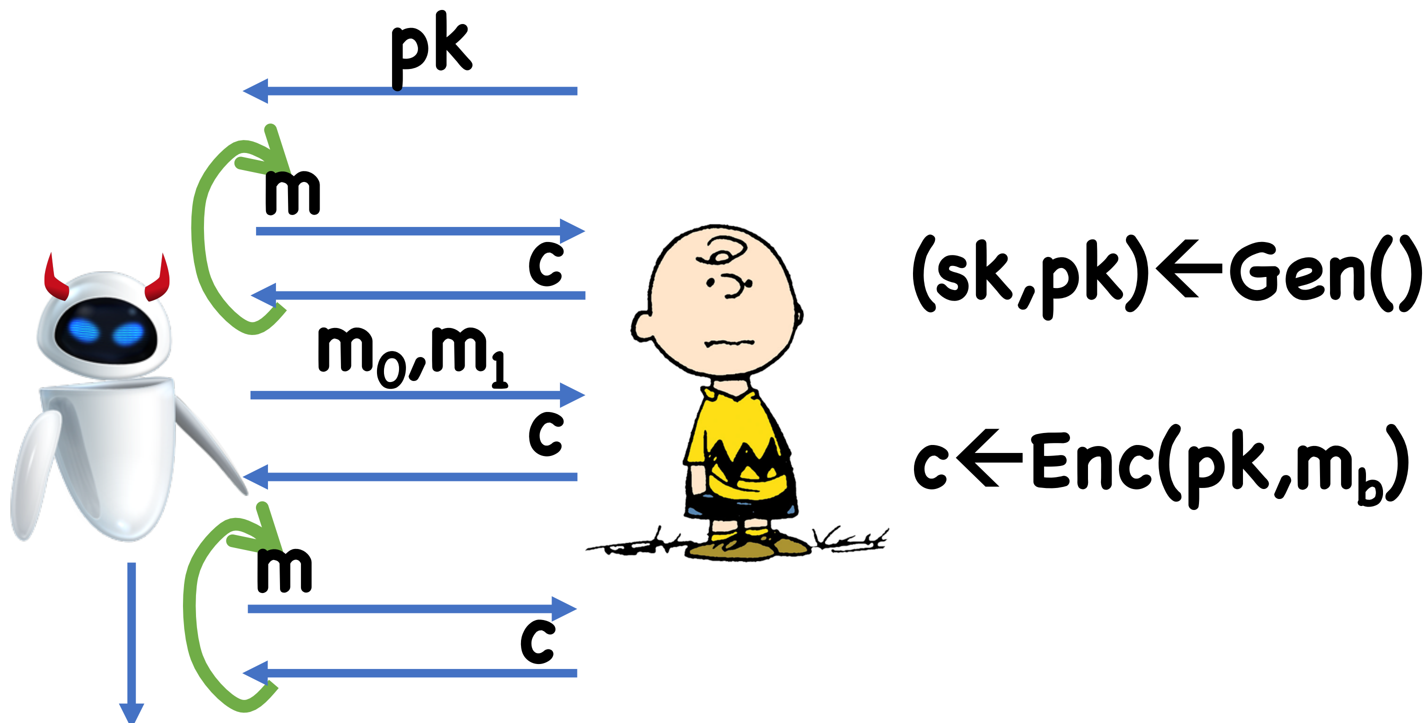
One-way Security



Semantic Security

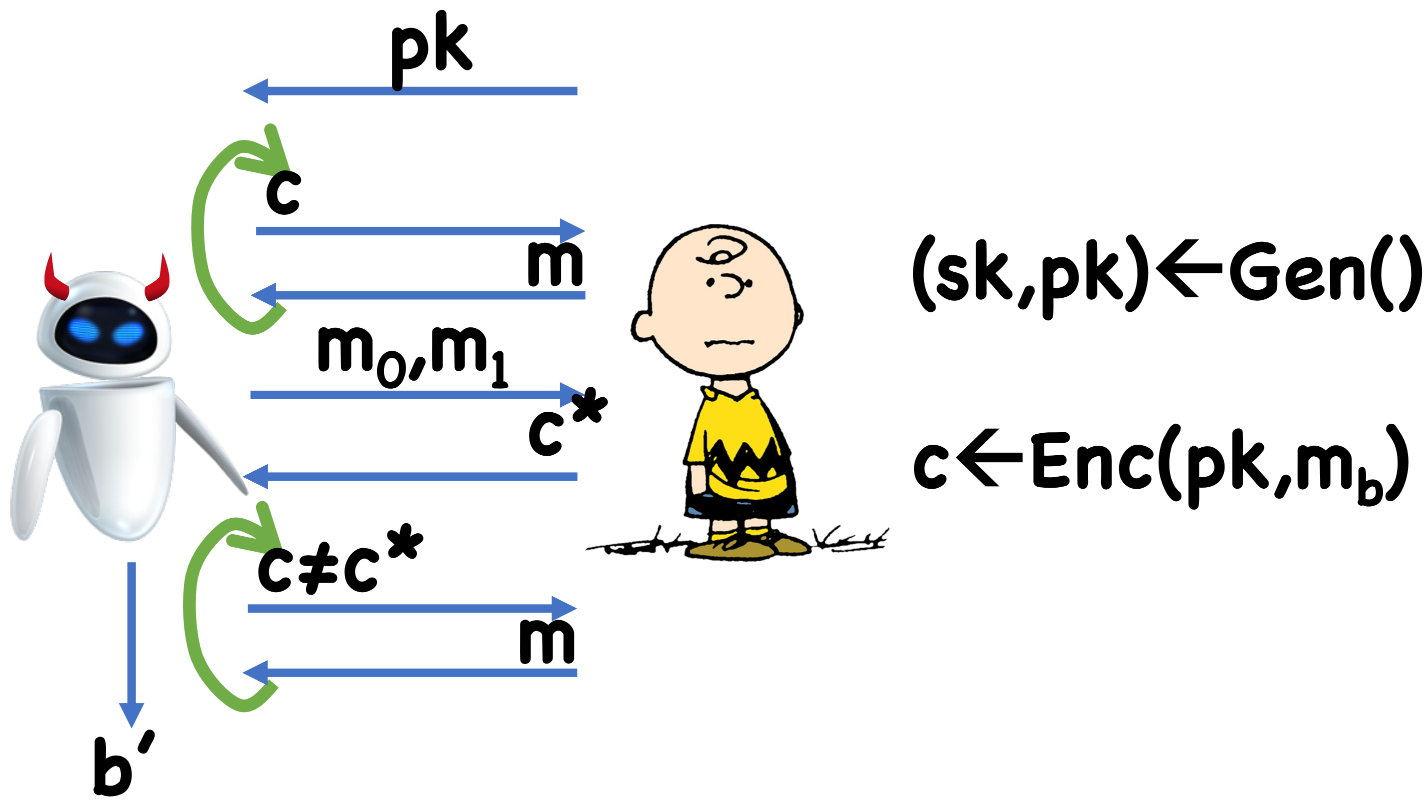


CPA Security



Theorem: An encryption scheme **(Gen,Enc,Dec)** is semantically secure if and only if it is CPA secure

CCA Security



One-way Encryption from RSA

Gen():

- Choose random primes **p,q**
- Let **N=pq**
- Choose **e,d** .s.t **ed=1 mod (p-1)(q-1)**
- Output **pk=(N,e), sk=(N,d)**

Enc(pk,m): Output $c = m^e \bmod N$

Dec(sk,c): Output $m' = c^d \bmod N$

Goldwasser-Micali

Gen():

- Choose random primes **p,q**
- Let **N=pq**
- Choose **x** a quadratic non-residue mod **p** and **q**
- Output **pk=(N,x), sk=(p,q)**

Enc(pk,m$\in${0,1}): $r \leftarrow \mathbb{Z}_N^*$, $c \leftarrow x^m r^2$ mod N

- If **m=0**, then c is a quadratic residue
- If **m=1**, then c is a non-residue

ElGamal

Group **G** of order **p**, generator **g**
Message space = **G**

Gen():
- Choose random $a \leftarrow \mathbb{Z}_p^*$, let $h \leftarrow g^a$
- $pk = h$, $sk = a$

Enc(pk,m∈{0,1}):
- $r \leftarrow \mathbb{Z}_p$
- $c = (g^r, h^r \times m)$

Dec?

Today

Trapdoor Permutations: abstracting RSA

CCA-secure Encryption in the ROM

Begin: digital signatures

Trapdoor Permutations

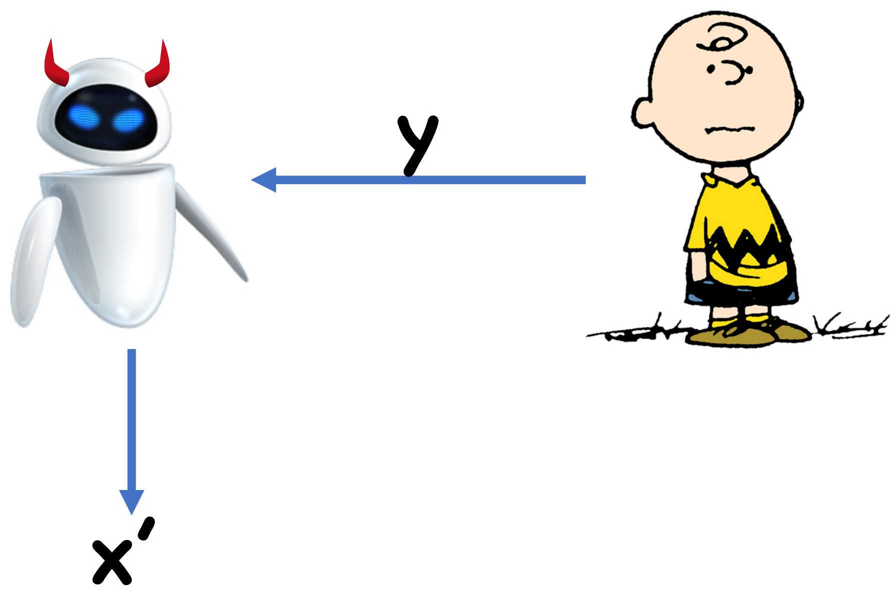
Domain $\mathbf{X}$

Gen(): outputs **(pk,sk)**
$\mathbf{F(pk, x \in X) = y \in X}$
$\mathbf{F^{-1}(sk, y) = x}$

Correctness:
$\mathbf{Pr[\ F^{-1}(sk,\ F(pk,\ x)) = x : (pk,sk) \leftarrow Gen()\ ] = 1}$

Correctness implies $\mathbf{F, F^{-1}}$ are deterministic, permutations

Trapdoor Permutation Security



(sk,pk)←Gen()
x←X
y←F(pk,x)

One-way Encryption from TDPs

$\mathbf{Gen_{Enc}() = Gen()}$

$\mathbf{Enc(pk,x) = F(pk, x)}$

$\mathbf{Dec(sk,c) = F^{-1}(sk, c)}$

Thus, TDPs are special case of one-way encryption where **Enc** is deterministic and $\mathbf{C = M}$

CPA-Secure Encryption from TDPs

Let **h** be a hardcore bit for the one-way function
$x \rightarrow F(pk,x)$

$\mathrm{Enc}(pk,b) = F(pk,r),\ h(r) \oplus b$

Constructing TDPs with hardcore bits?

- $F'(pk,\ (r,x)\ ) = (r\ ,\ F(pk,x))$
- $h(r,x) = r \oplus b$

Trapdoor Permutations from RSA

Gen():

- Choose random primes $\mathbf{p,q}$
- Let $\mathbf{N=pq}$
- Choose $\mathbf{e,d}$ .s.t $\mathbf{ed=1 \bmod (p-1)(q-1)}$
- Output $\mathbf{pk=(N,e),\ sk=(N,d)}$

F(pk,x): Output $\mathbf{y = x^e \bmod N}$

$\mathbf{F^{-1}}$(sk,c): Output $\mathbf{x = y^d \bmod N}$

Caveats

RSA is not a true TDP as defined

- Why???
- What’s the domain?

Nonetheless, distinction is not crucial to most applications

Other TDPs?

For long time, none known

- Still interesting object:
    - Useful abstraction in protocol design
    - Maybe more will be discovered…

Using obfuscation:

- Let P be a PRP
- $\mathbf{sk = k,\ pk = Obf(\ P(k, \cdot\ )\ )}$

Relaxation: Injective Trapdoor Functions

Domain $\mathbf{X}$, range $\mathbf{Y}$

$\mathbf{Gen()}$: outputs $\mathbf{(pk,sk)}$
$\mathbf{F(pk,x \in X) = y \in Y}$, deterministic
$\mathbf{F^{-1}(sk,y) = x}$

Correctness:
$\mathbf{Pr[\ F^{-1}(sk,\ F(pk,\ x)) = x : (pk,sk) \leftarrow Gen()\ ] = 1}$

Correctness implies $\mathbf{F}$ is injective

Injective TDFs from DH

Notation:

Let $\mathbf{A} \in \mathbb{Z}_p^{\mathbf{n\times n}}$

$\mathbf{g^A \in G^{n\times n}, \ (g^A)_{i,j} := g^{A_{i,j}}}$

Let $\mathbf{H} \in \mathbf{G^{n\times n}}, \ \mathbf{v} \in \mathbb{Z}_p^{\mathbf{n}}$

$\mathbf{H^v \in G^n, \ (H^v)_i := \Pi_j \ H_{i,j}{}^{v_j}}$

Note: $\mathbf{((g^A)^v)_i = \Pi_j \ g^{A_{i,j} v_j} = g^{(A\cdot v)_i}}$, so $\mathbf{(g^A)^v = g^{A\cdot v}}$

Injective TDFs from DH

Notation:

Let $\mathbf{h} \in \mathbf{G}^{\mathbf{n}}$, $\mathbf{A} \in \mathbb{Z}_{\mathbf{q}}^{\mathbf{n \times n}}$

$^{\mathbf{A}}\mathbf{h} \in \mathbf{G}^{\mathbf{n}}$, $(^{\mathbf{A}}\mathbf{h})_{\mathbf{i}} := \prod_{\mathbf{j}} \mathbf{h}_{\mathbf{j}}^{\mathbf{A}_{\mathbf{i,j}}}$

$^{\mathbf{A}}(\mathbf{g}^{\mathbf{v}})$?

$(^{\mathbf{A}}(\mathbf{g}^{\mathbf{v}}))_{\mathbf{i}} = \prod_{\mathbf{j}} \mathbf{g}^{\mathbf{A}_{\mathbf{i,j}}\mathbf{v}_{\mathbf{j}}} = \mathbf{g}^{(\mathbf{A \cdot v})_{\mathbf{i}}}$, so $^{\mathbf{A}}(\mathbf{g}^{\mathbf{v}}) = \mathbf{g}^{\mathbf{A \cdot v}}$

Injective TDFs from DH

First Attempt:

Gen(): choose random $\mathbf{A} \leftarrow \mathbb{Z}_q^{n \times n}$

$\mathbf{sk} = \mathbf{A},\ \mathbf{pk} = \mathbf{H} = g^{\mathbf{A}}$

$\mathbf{F(pk,\ x} \in \mathbb{Z}_q^{\mathbf{n}}\mathbf{)} = \mathbf{H}^{\mathbf{x}}\ (= g^{\mathbf{A} \cdot \mathbf{x}})$

$\mathbf{F^{-1}(sk,\ h)}$: $\mathbf{y} \leftarrow {}^{\mathbf{A}^{-1}}\mathbf{h}\ (= g^{\mathbf{A}^{-1} \cdot \mathbf{A} \cdot \mathbf{x}} = g^{\mathbf{x}})$

Then Dlog?????

Injective TDFs from DH

Theorem: If DDH holds, then (Gen,F,F^{-1}) is an injective TDF

Gen(): choose random $A \leftarrow \mathbb{Z}_q^{n\times n}$

$sk = A,\ pk = H = g^A$

$F(pk,\ x \in \{0,1\}^n) = H^x\ (= g^{A\cdot x})$

$F^{-1}(sk,\ h)$: $y \leftarrow {}^{A^{-1}}h\ (= g^{A^{-1}\cdot A\cdot x} = g^x\ )$

Then Dlog each component to recover x

Injective TDFs

Known constructions from most number theoretic problems

Useful in many cases where TDPs are used (but not all)

CCA Secure PKE from TDPs

Let $(\mathsf{Enc}_{\mathsf{SKE}},\mathsf{Dec}_{\mathsf{SKE}})$ be a CCA-secure secret key encryption scheme.

Let $(\mathsf{Gen},\mathsf{F},\mathsf{F}^{-1})$ be a TDP

Let H be a hash function (we'll pretend it's a random oracle)

CCA Secure PKE from TDPs

$\mathbf{Gen_{PKE}() = Gen()}$

$\mathbf{Enc_{PKE}(pk, m)}$:

- Choose random $\mathbf{r}$
- Let $\mathbf{c_0 \leftarrow F(pk,r)}$
- Let $\mathbf{c_1 \leftarrow Enc_{SKE}(H(r), m)}$
- Output $\mathbf{(c_0,c_1)}$

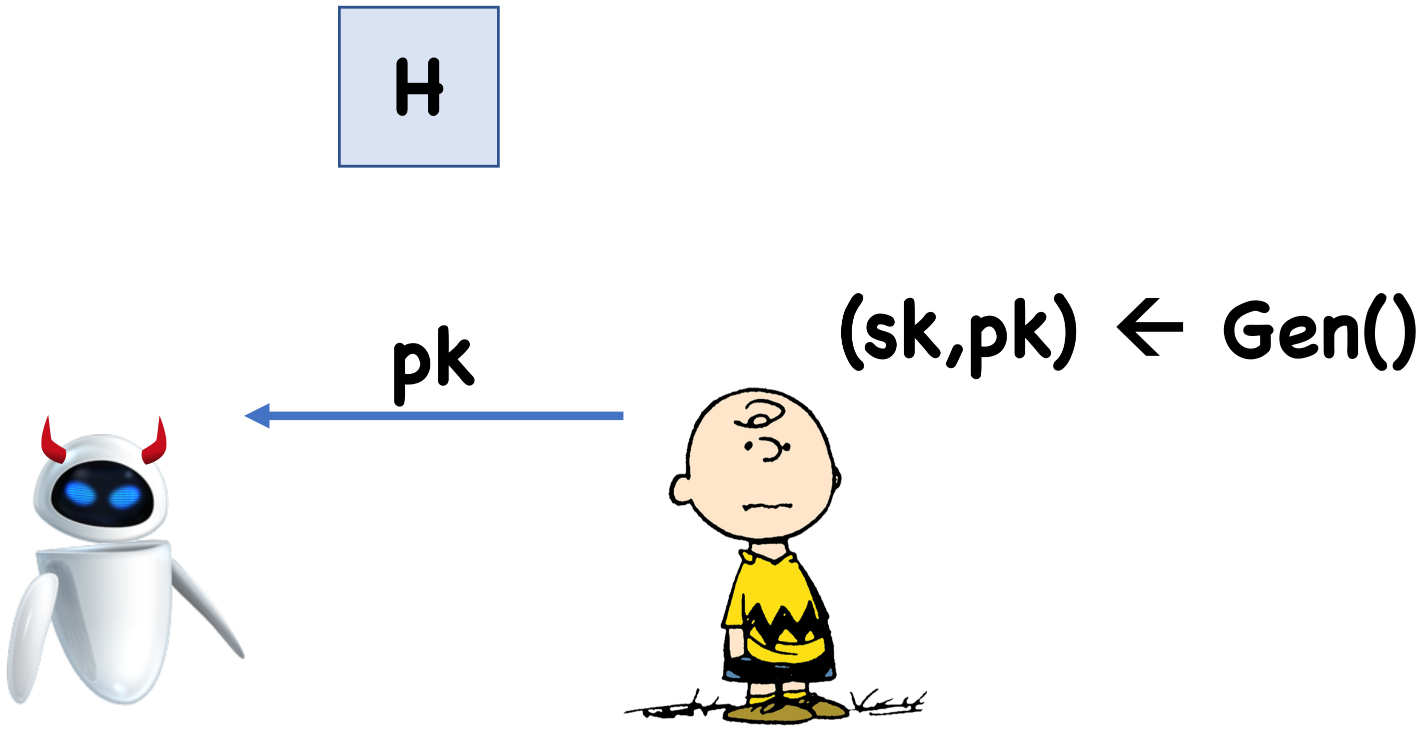
$\mathbf{Dec_{PKE}(sk, (c_0, c_1))}$:

- **Let** $\mathbf{r \leftarrow F^{-1}(sk, c_0)}$
- **Let** $\mathbf{m \leftarrow Dec_{SKE}(H(r), c_1)}$

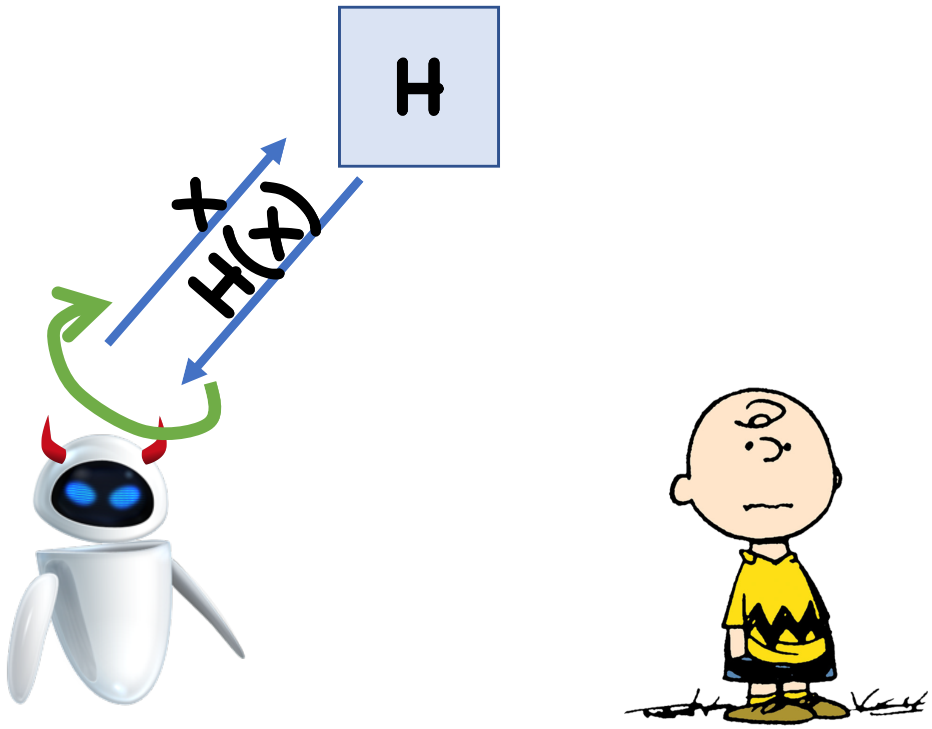
CCA Secure PKE from TDPs

Theorem: If $(\mathsf{Enc}_{SKE},\mathsf{Dec}_{SKE})$ is a CCA-secure secret key encryption scheme, $(\mathsf{Gen},\mathsf{F},\mathsf{F}^{-1})$ is a TDP, and H is modeled as a random oracle, then $(\mathsf{Gen}_{PKE},\mathsf{Enc}_{PKE},\mathsf{Dec}_{PKE})$ is a CCA secure public key encryption scheme

Proof



Proof



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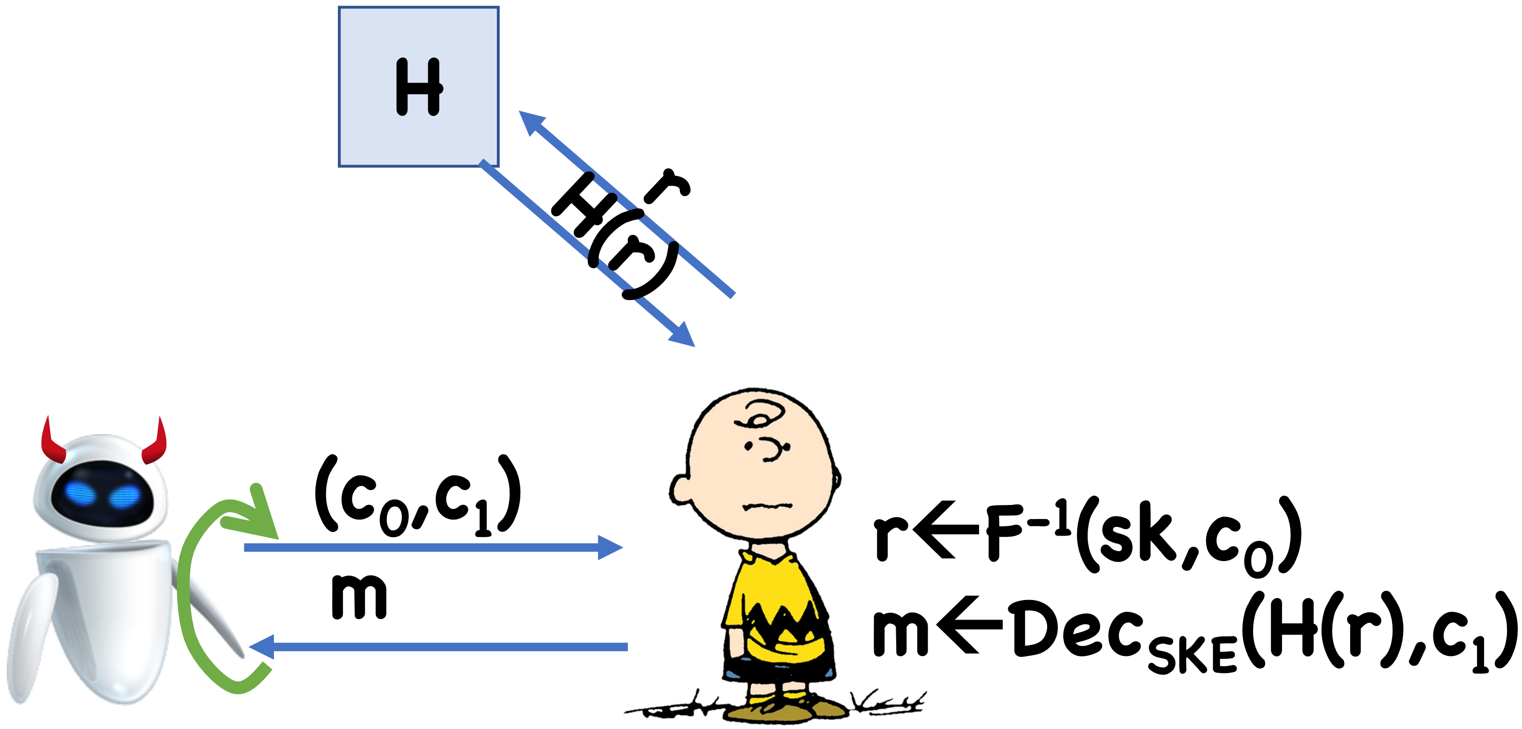
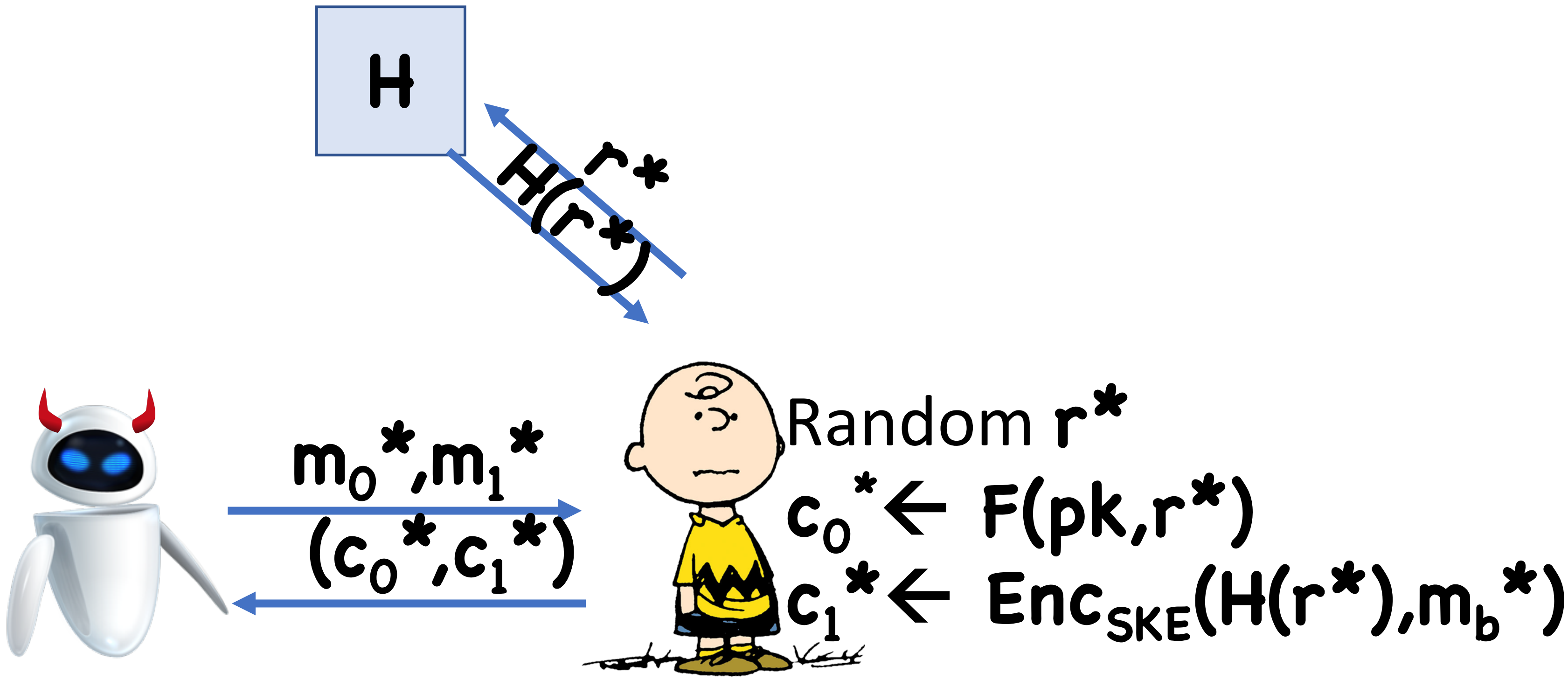
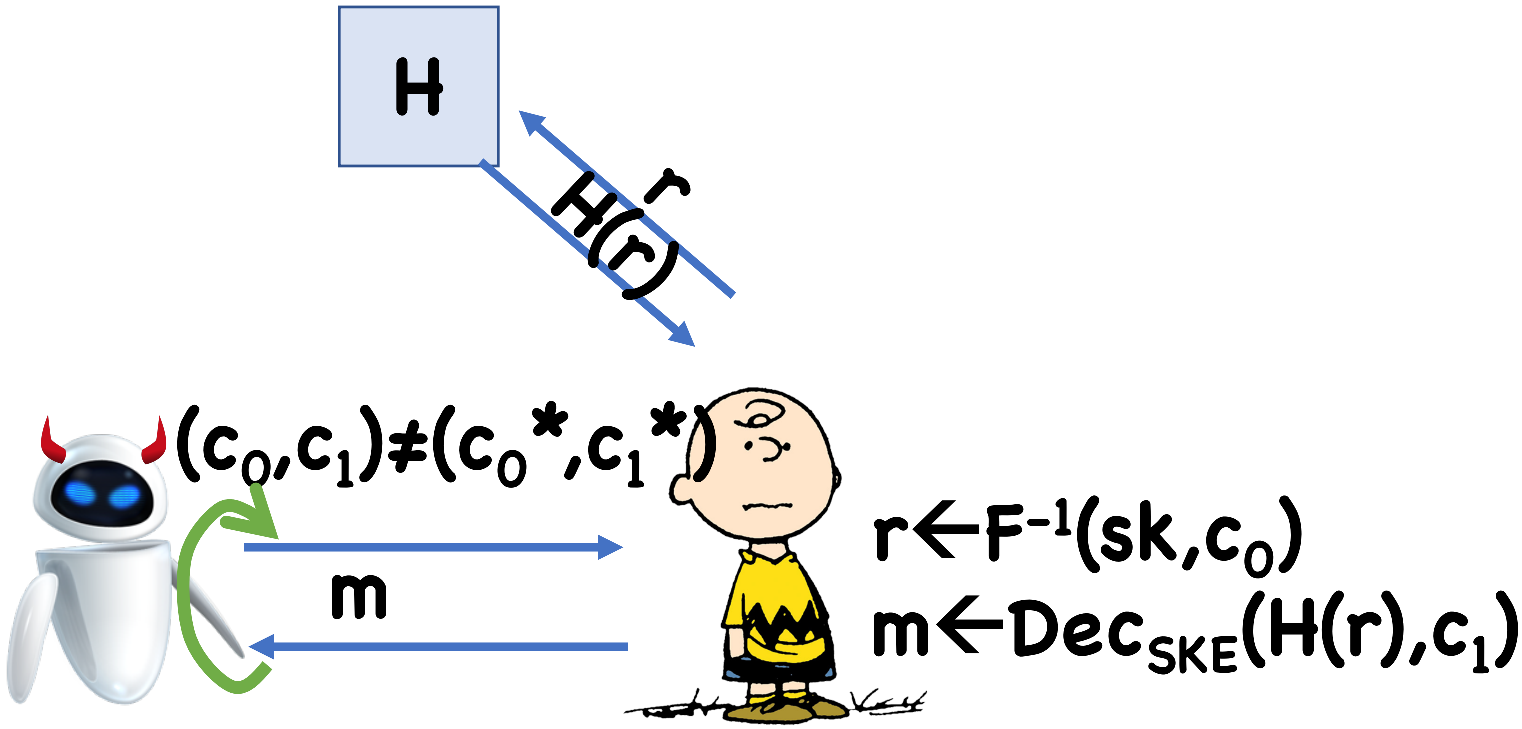


Proof



Proof



Proof

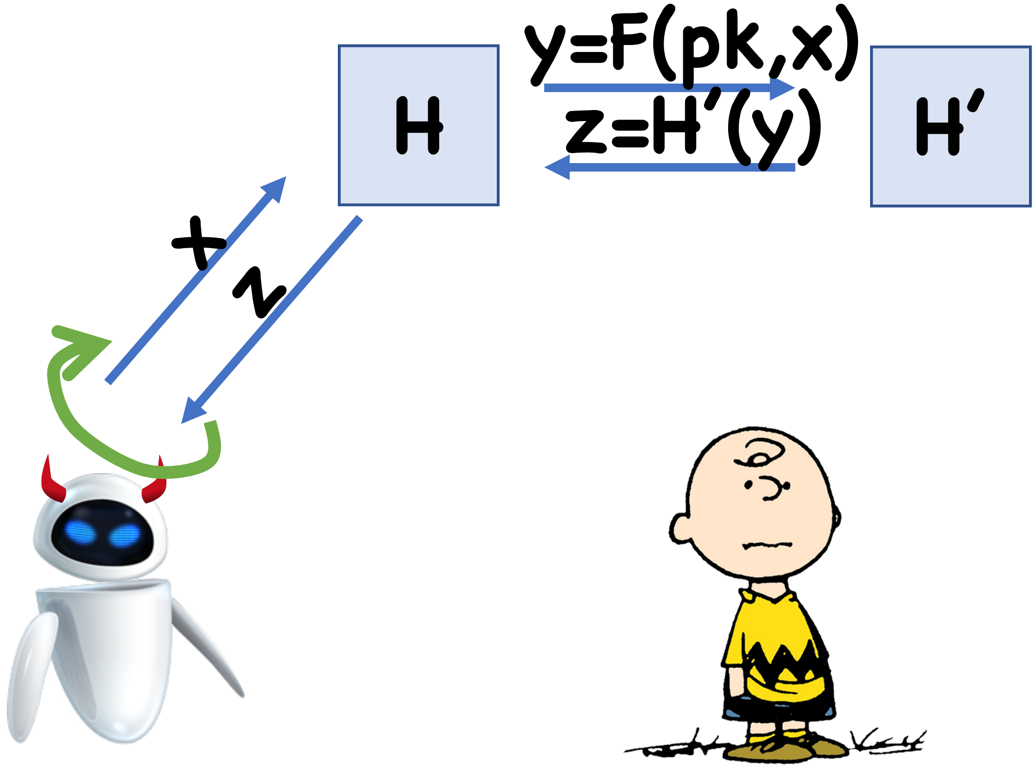
Step 1: sample H as follows:

- Choose a random function H'
- Let $H(x) = H'(\ F(pk,\ x)\ )$

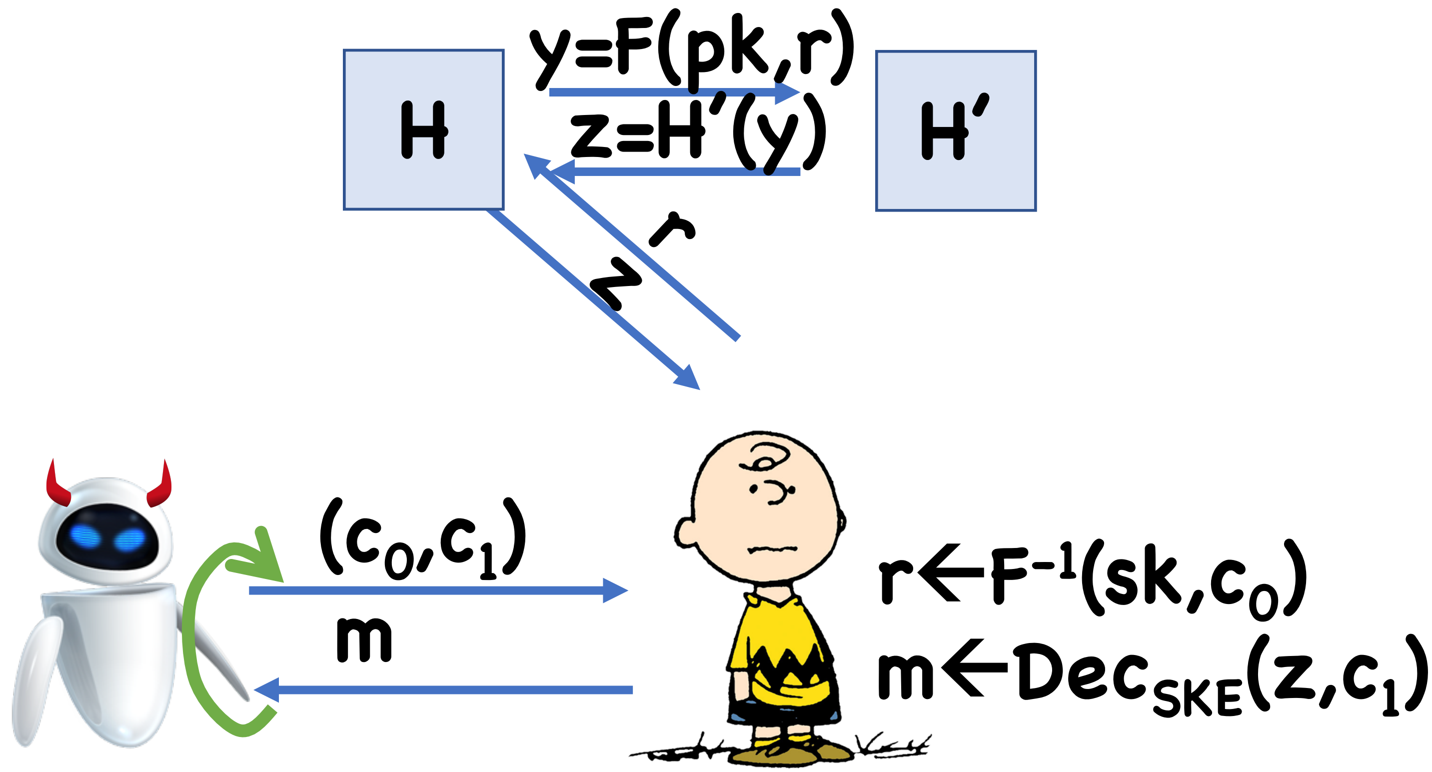
Since $F(pk, \cdot\ )$ is a permutation, all outputs of $H(x)$ are independent and uniform

Therefore, $H(x)$ is still a random oracle

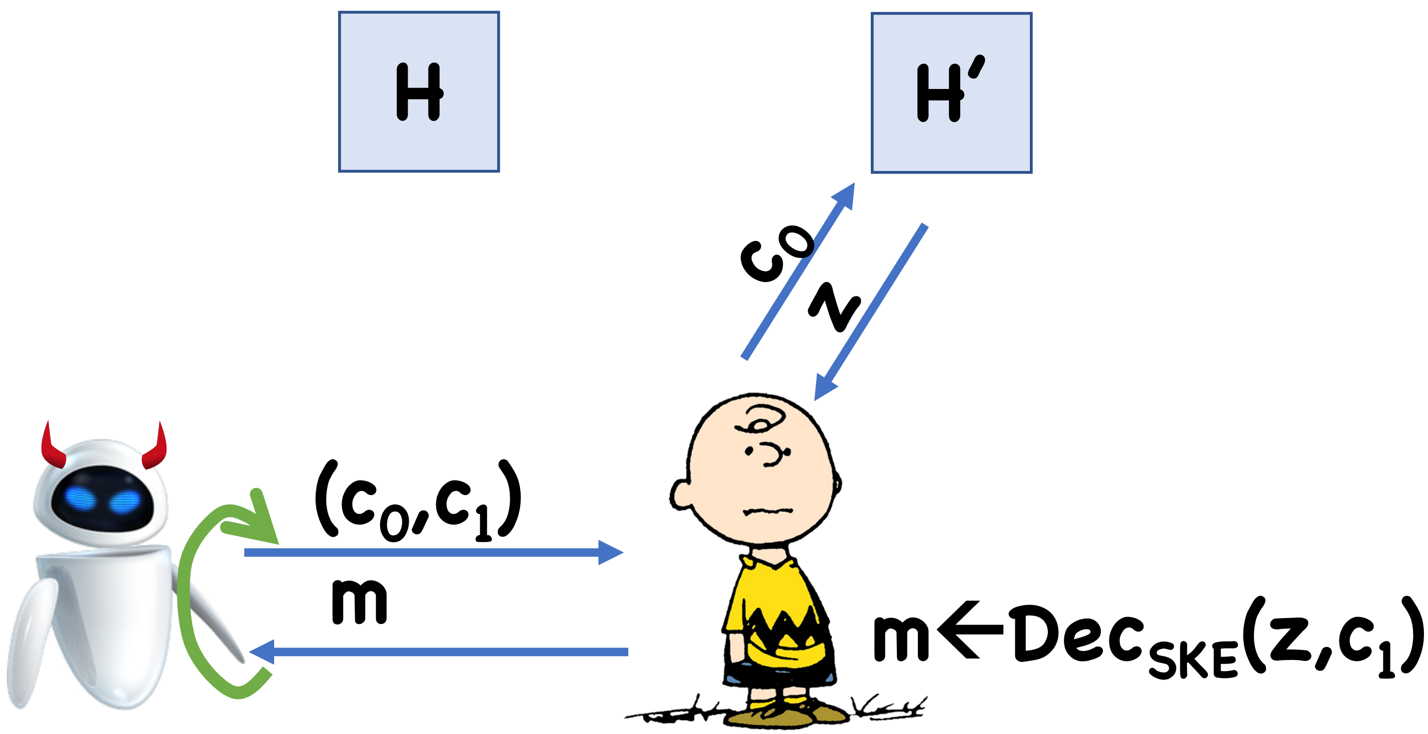
Proof



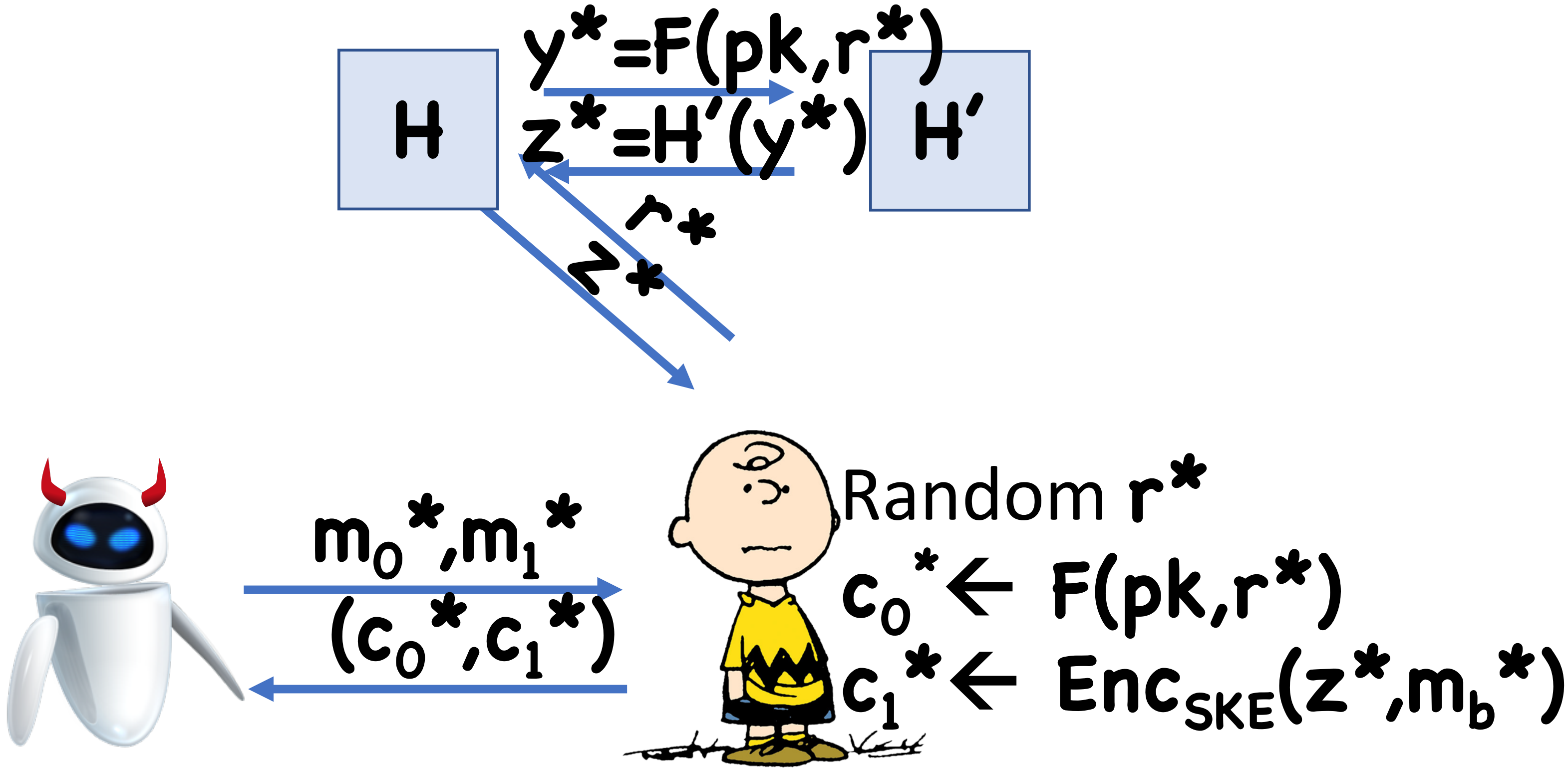
Proof



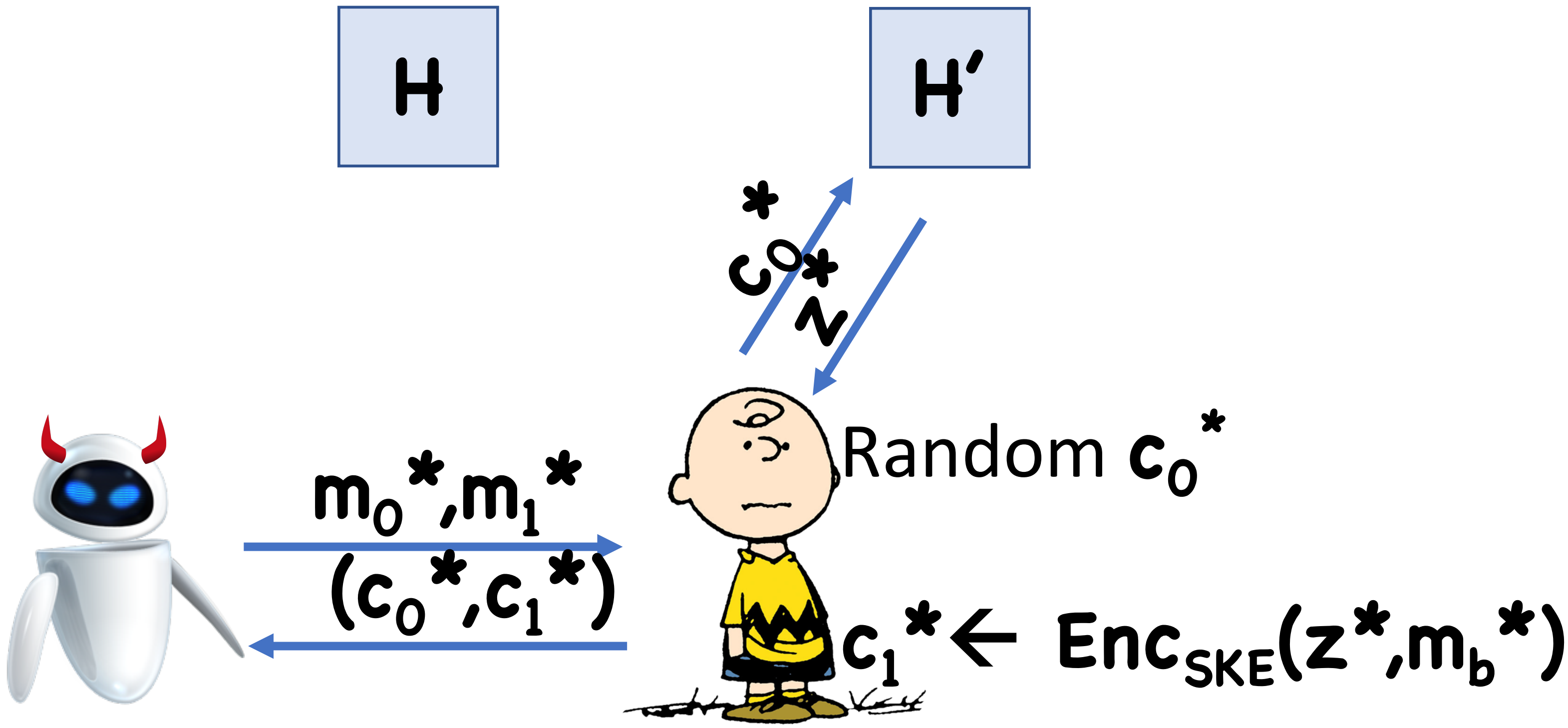
Proof



Proof



Proof



Observation: now Charlie doesn't need **sk** to run experiment

Proof

Consider two cases:

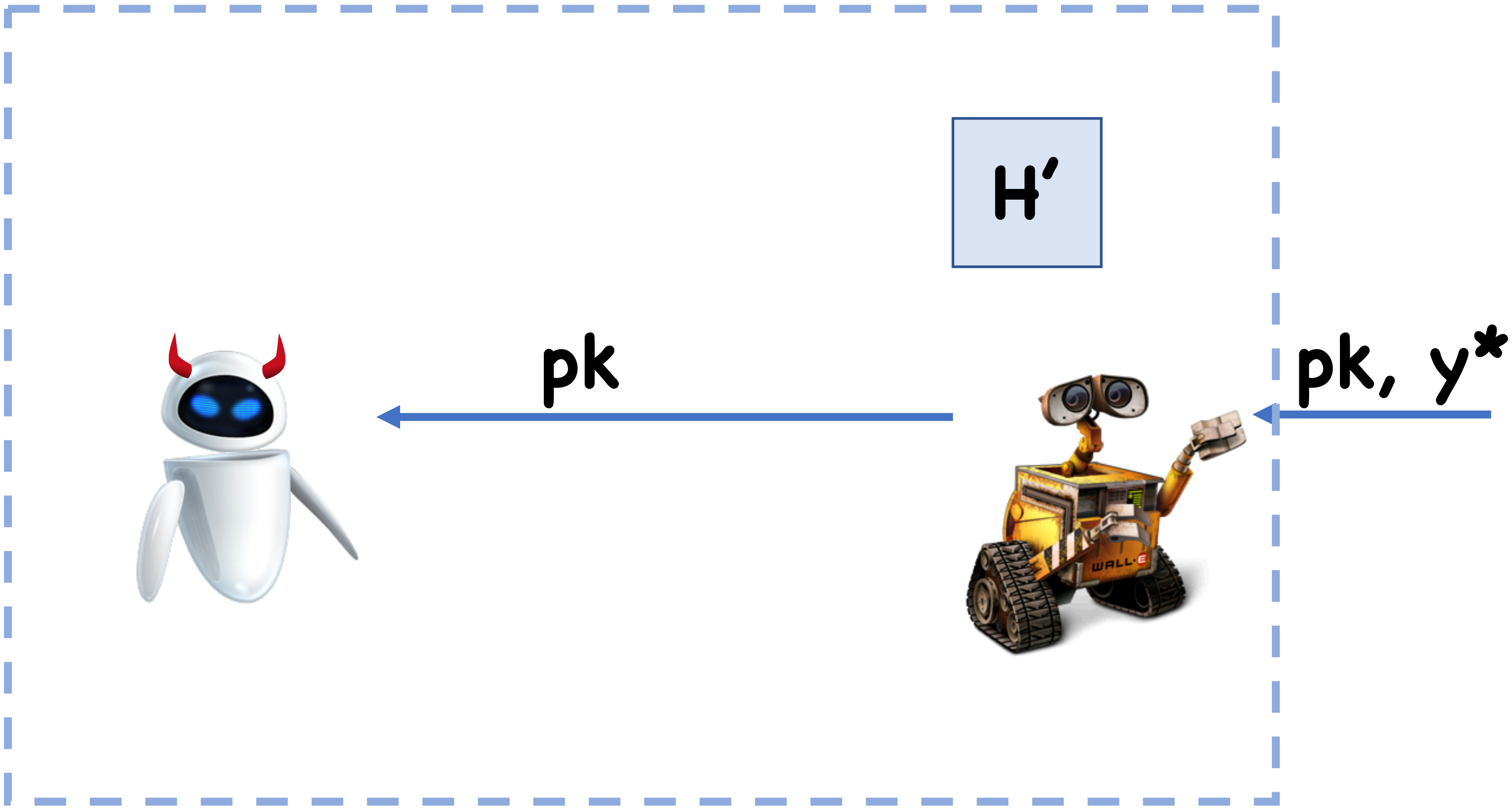
Case 1: adversary makes a RO query to $\mathbf{H}$ on

$$\mathbf{r^* = F^{-1}(sk,\ c_0^*)}$$

Case 2: adversary never makes a RO query on $\mathbf{r^*}$

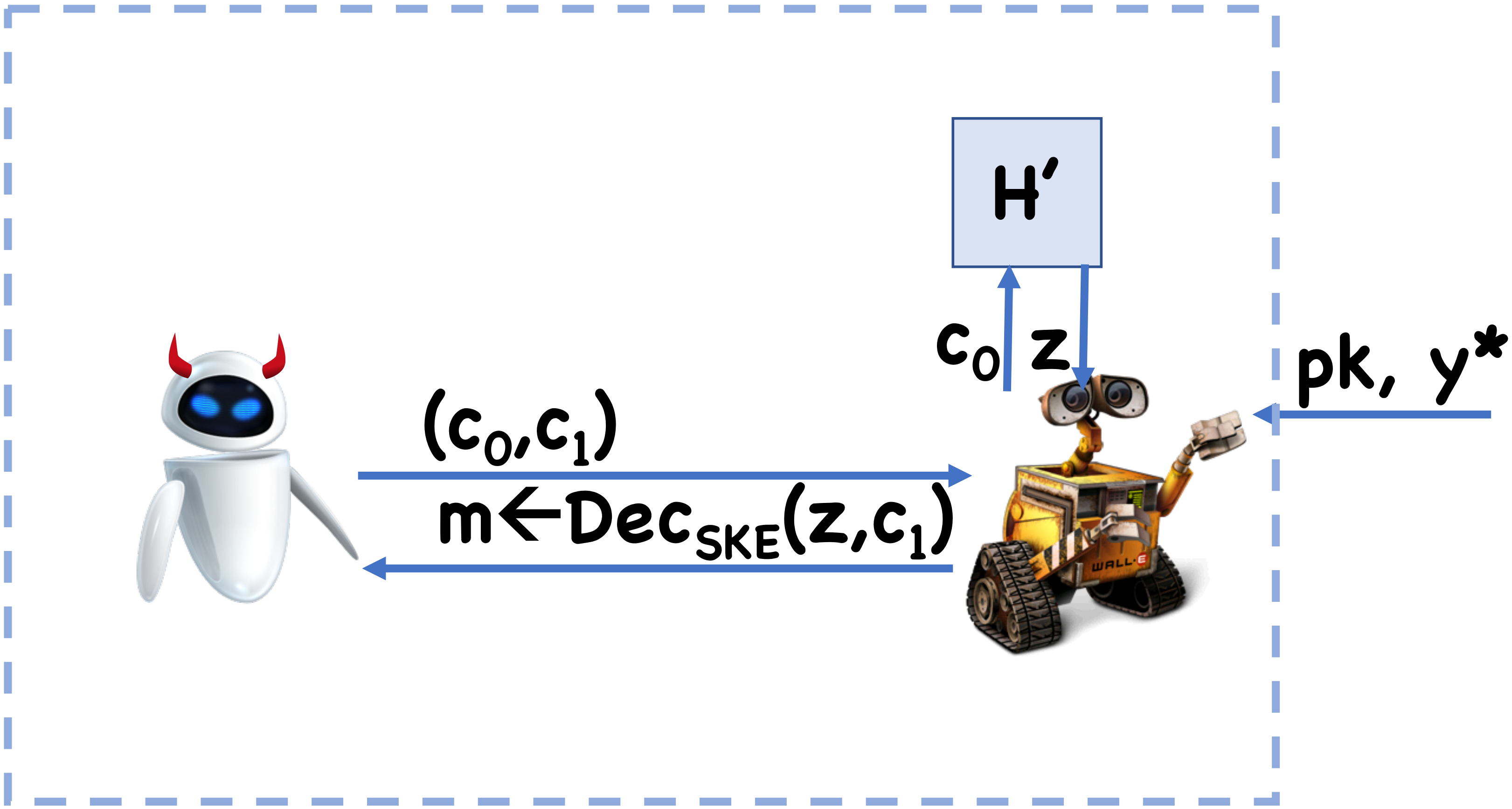
Proof

Case 1: construct TDP adversary 



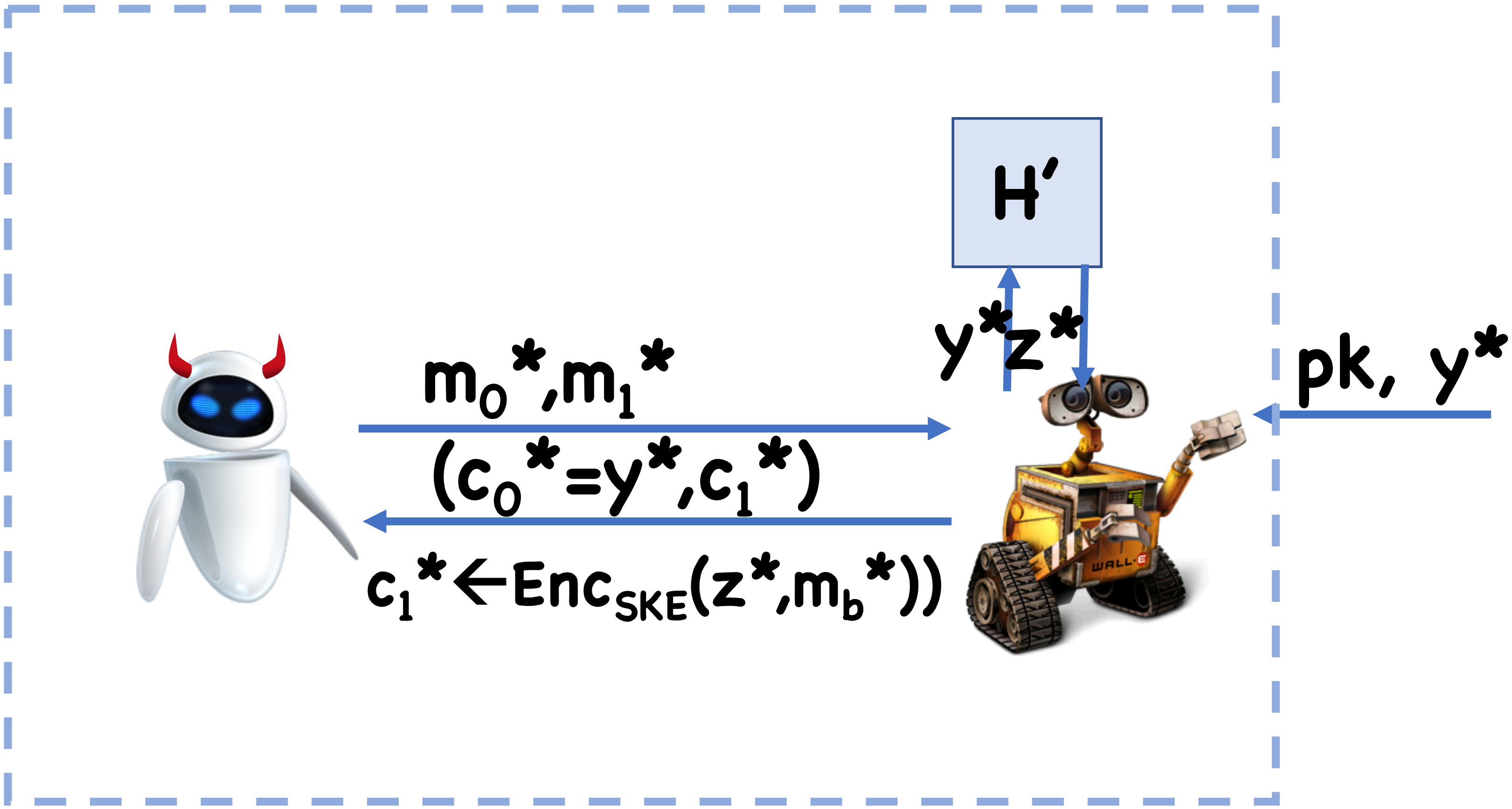
Proof

Case 1: construct TDP adversary 



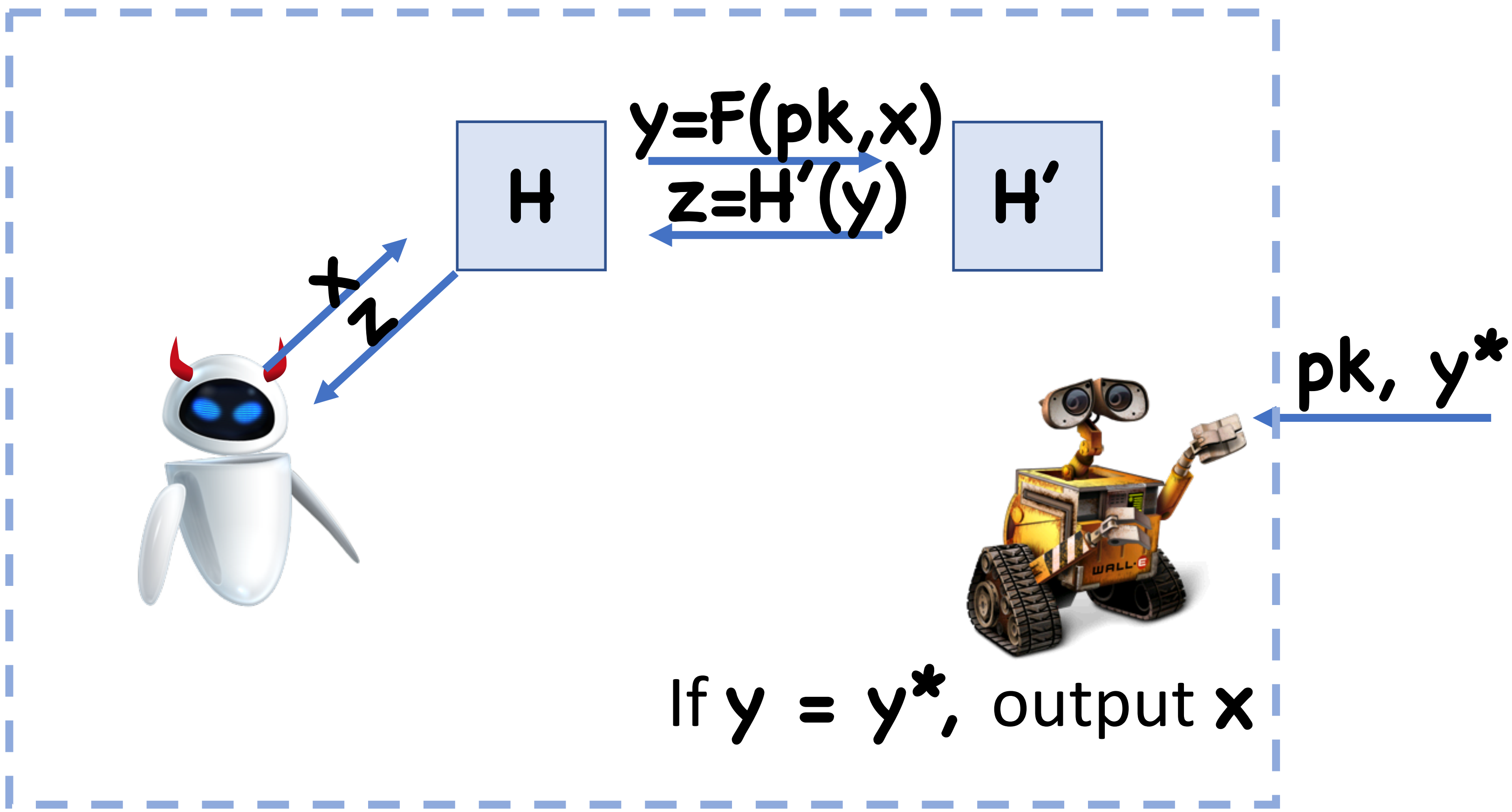
Proof

Case 1: construct TDP adversary 



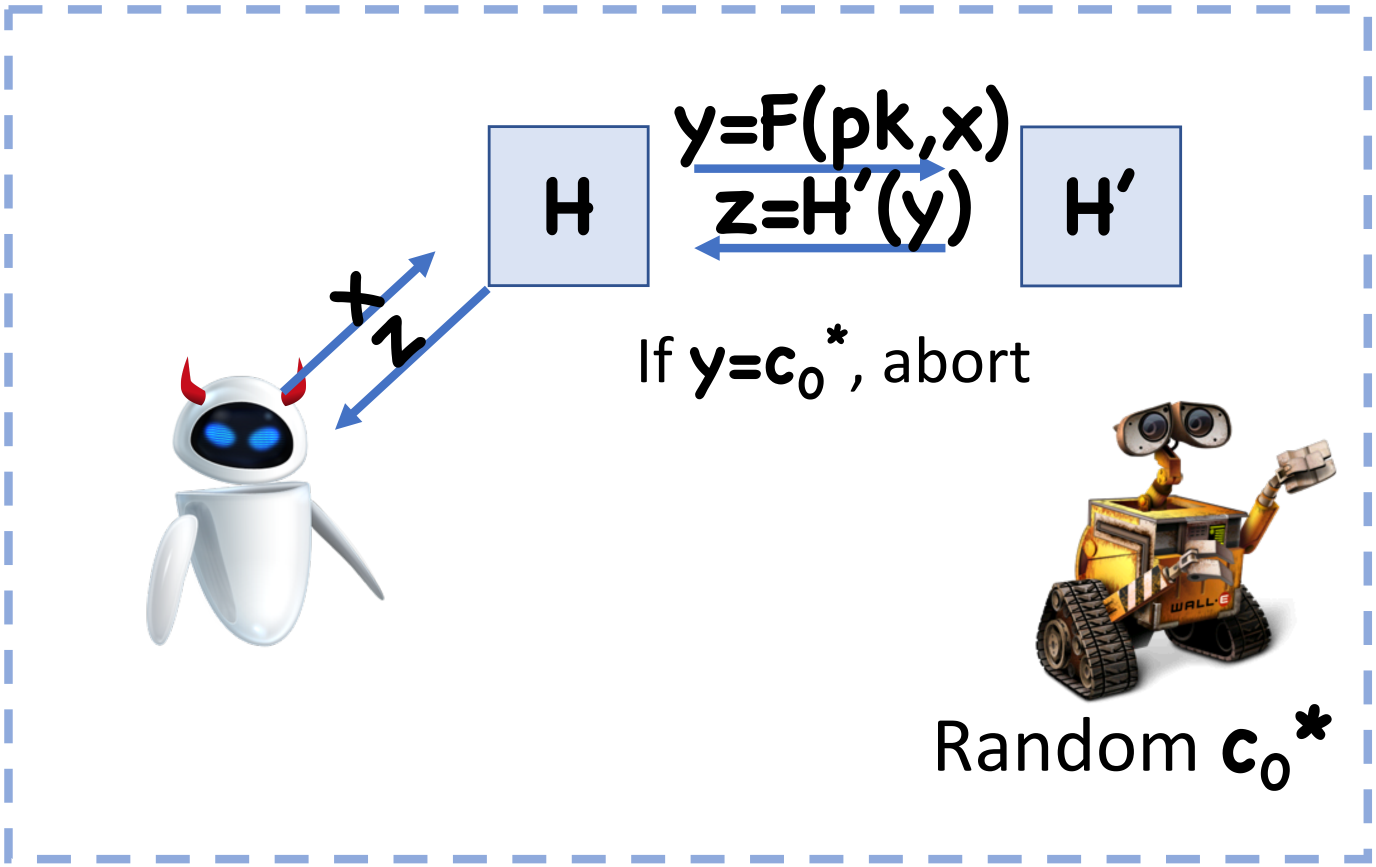
Proof

Case 1: construct TDP adversary 

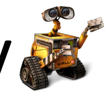


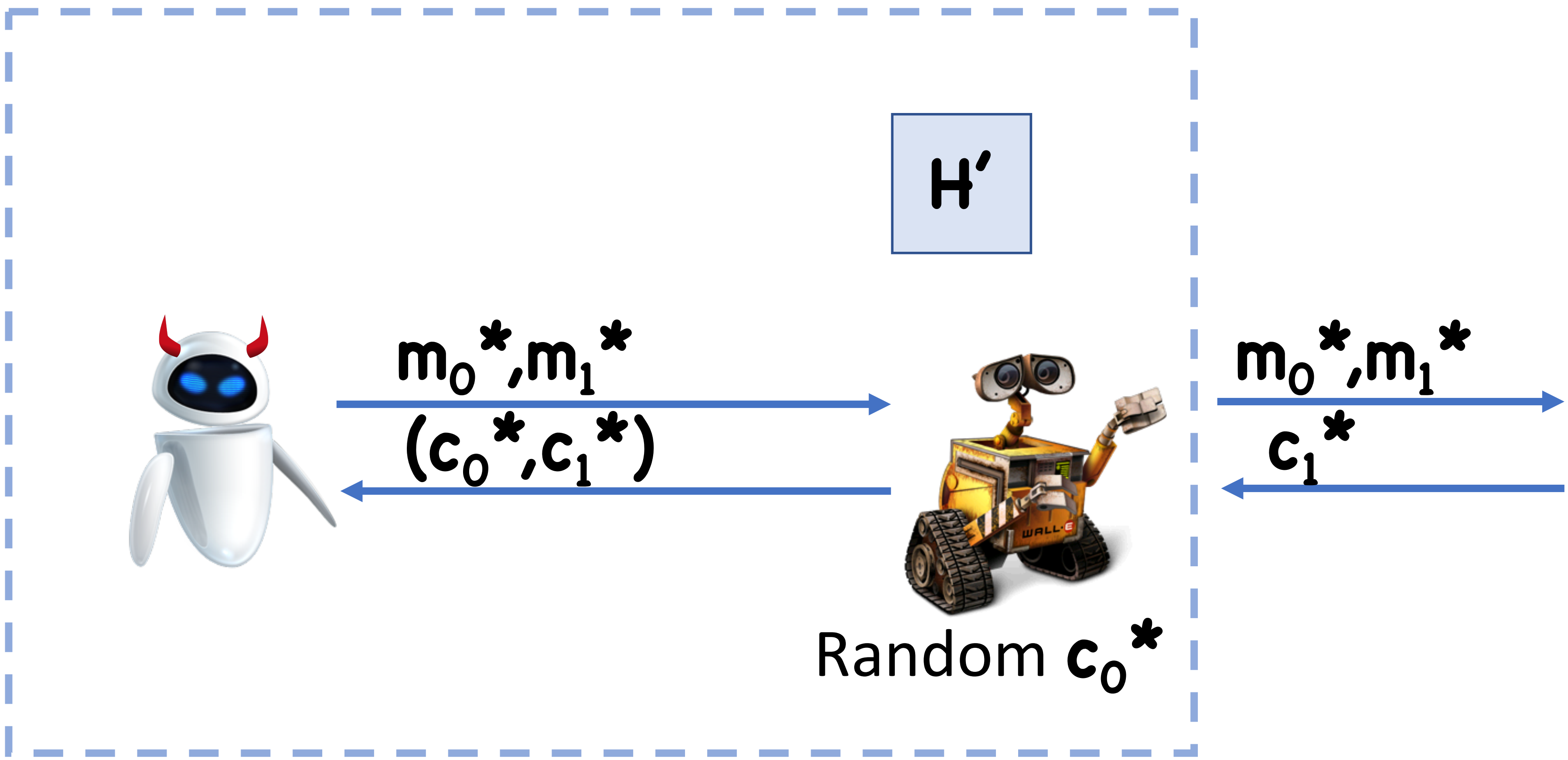
Proof

Case 2: construct **Enc**$_{\mathbf{SKE}}$ adversary

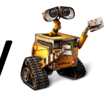


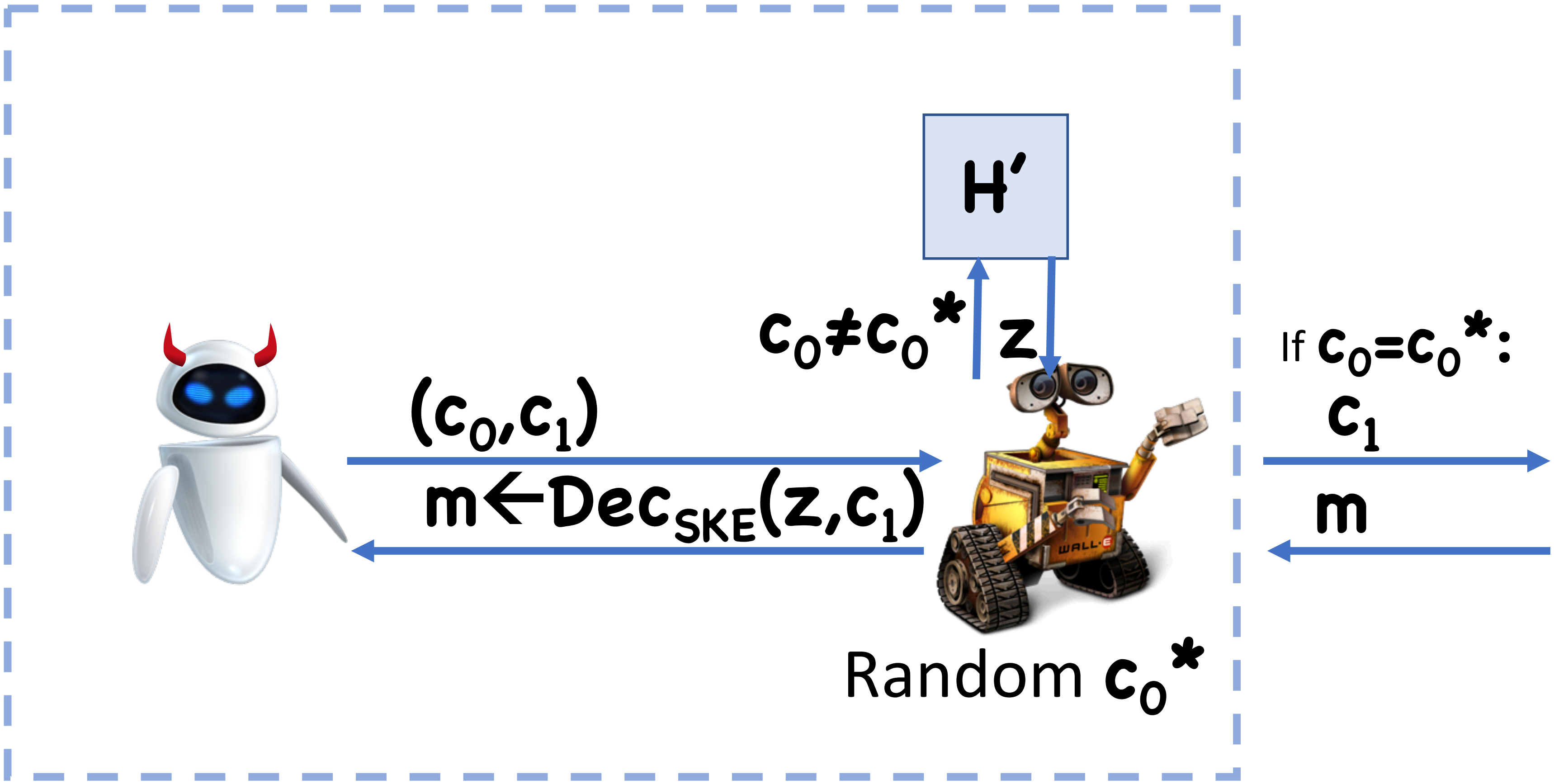
Proof

Case 2: construct $\mathbf{Enc}_{\mathbf{SKE}}$ adversary 

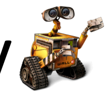


Proof

Case 2: construct $\mathbf{Enc_{SKE}}$ adversary 



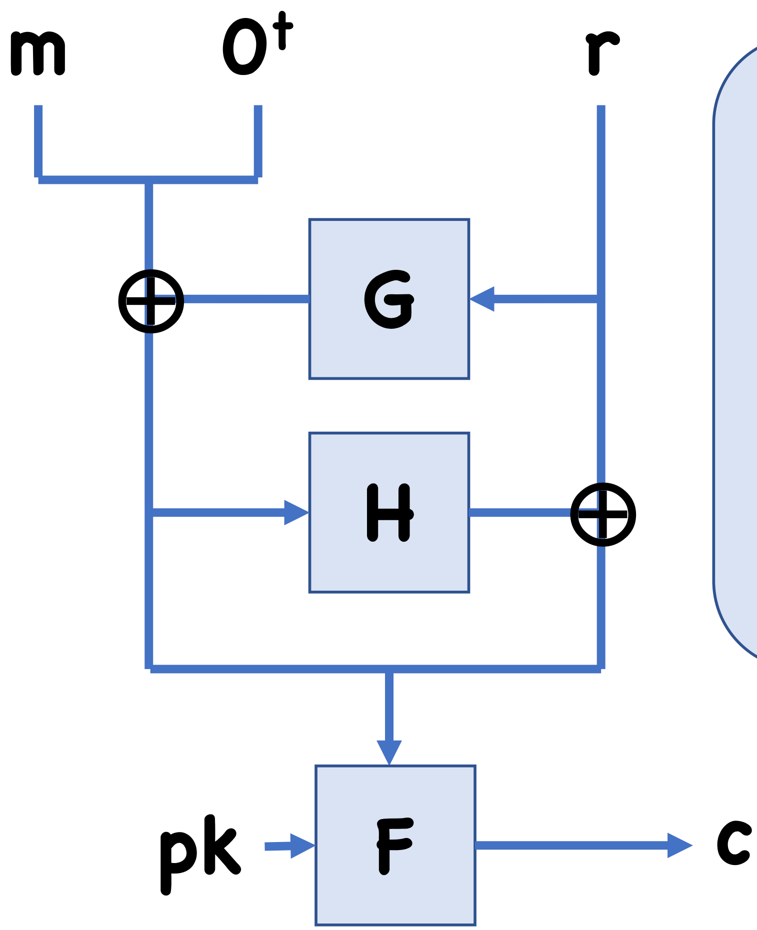
Proof

Case 2: construct $\mathsf{Enc}_{\mathsf{SKE}}$ adversary 

Analysis:

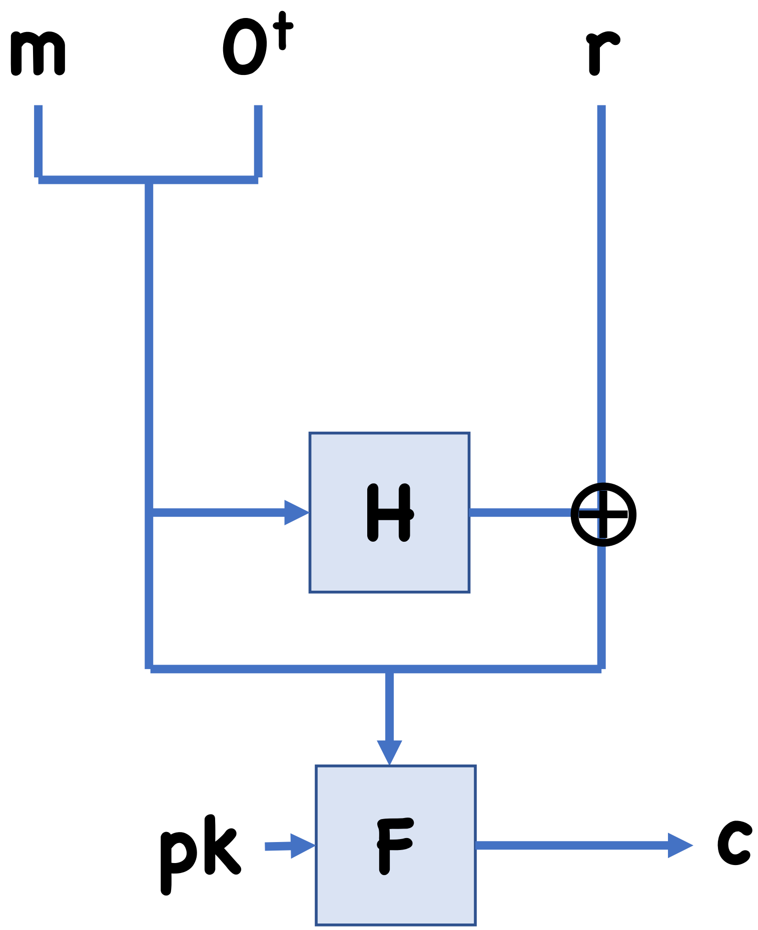
- Effectively set $\mathsf{H'}(\mathsf{c}_0^*) = \mathsf{k}$, where k is (unknown) challenger key
- Answers all queries correctly, provided adversary never queries RO on c_0^*
- Therefore, breaks security of $\mathsf{Enc}_{\mathsf{SKE}}$ in case 2

OAEP



Theorem: For RSA TDP, if **G,H** are modeled as a random oracles, then **(Gen$_{PKE}$,Enc$_{PKE}$,Dec$_{PKE}$)** is a CCA secure public key encryption scheme

Insecure OAEP Variants

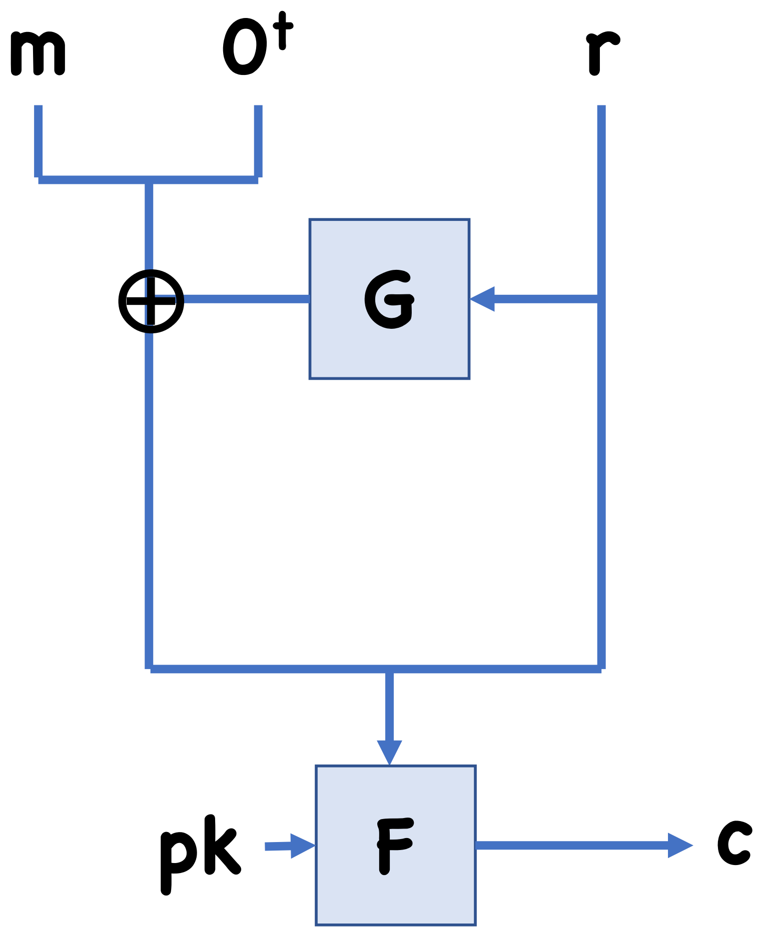


$c = F(pk, (m, 0^t, y))$

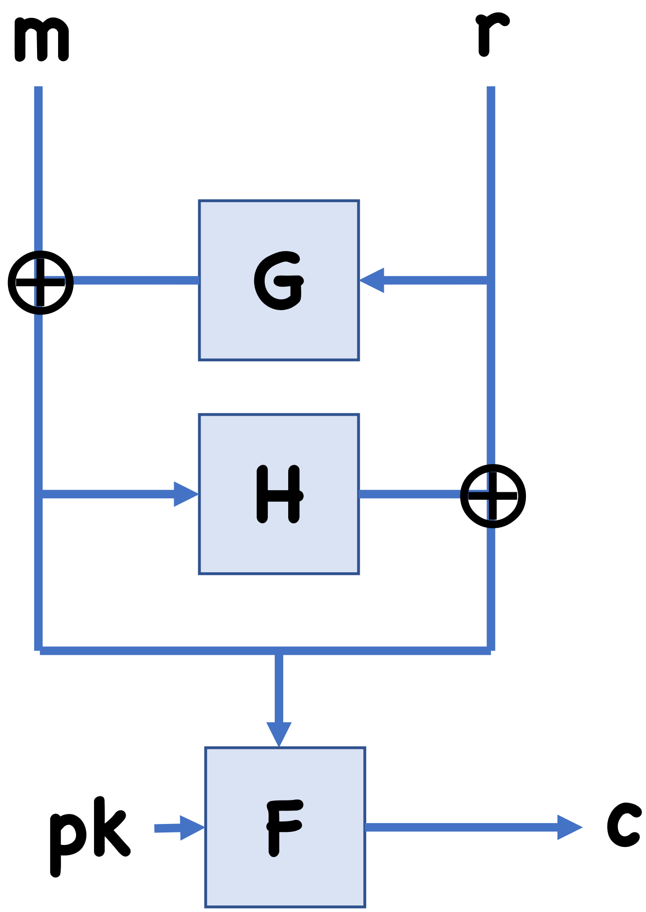
May contain **m** in the clear

- $F(pk, (m,x,y))$
  $= (m, F(pk, (x,y)))$

Insecure OAEP Variants



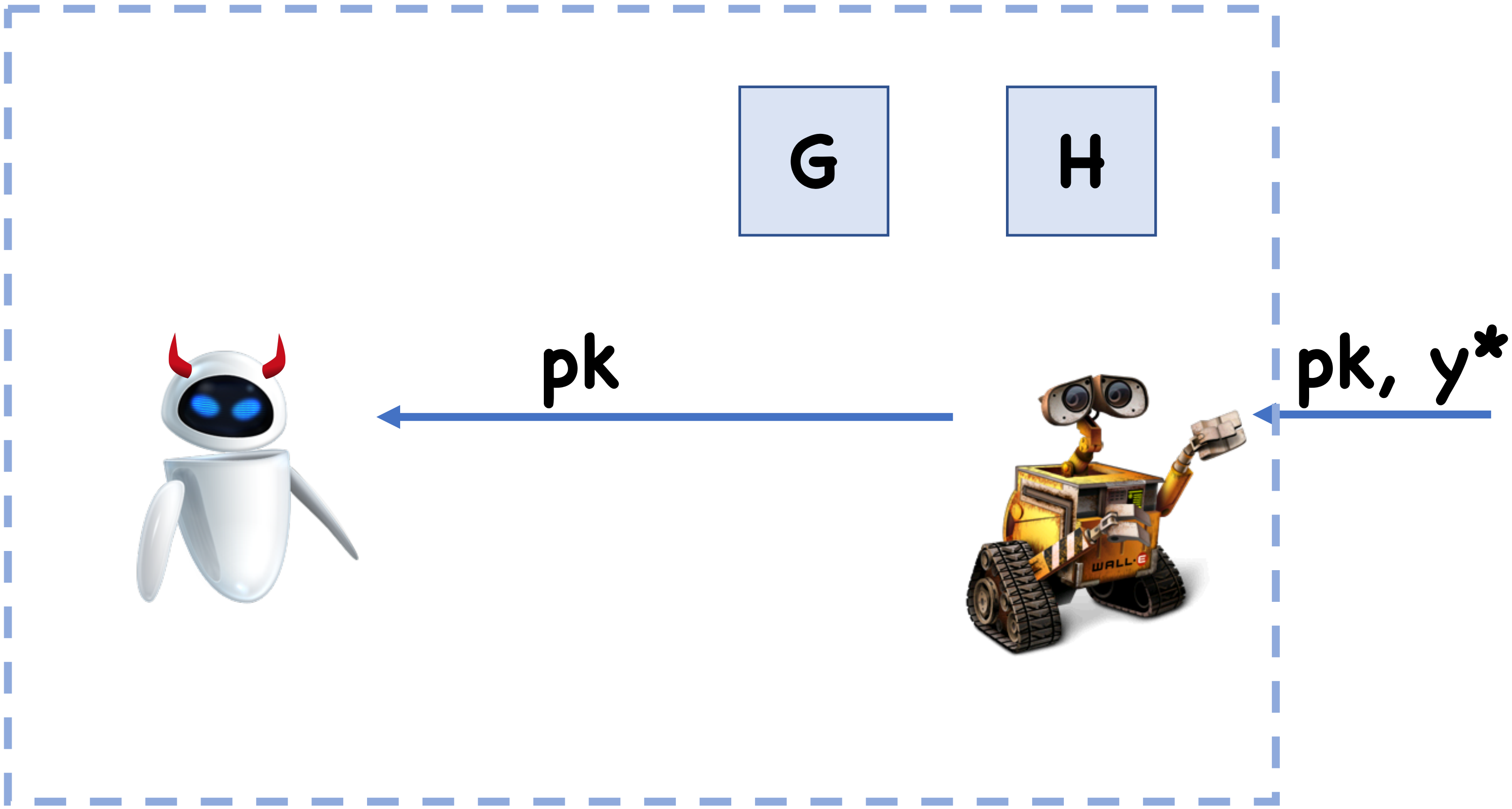
Why padding?



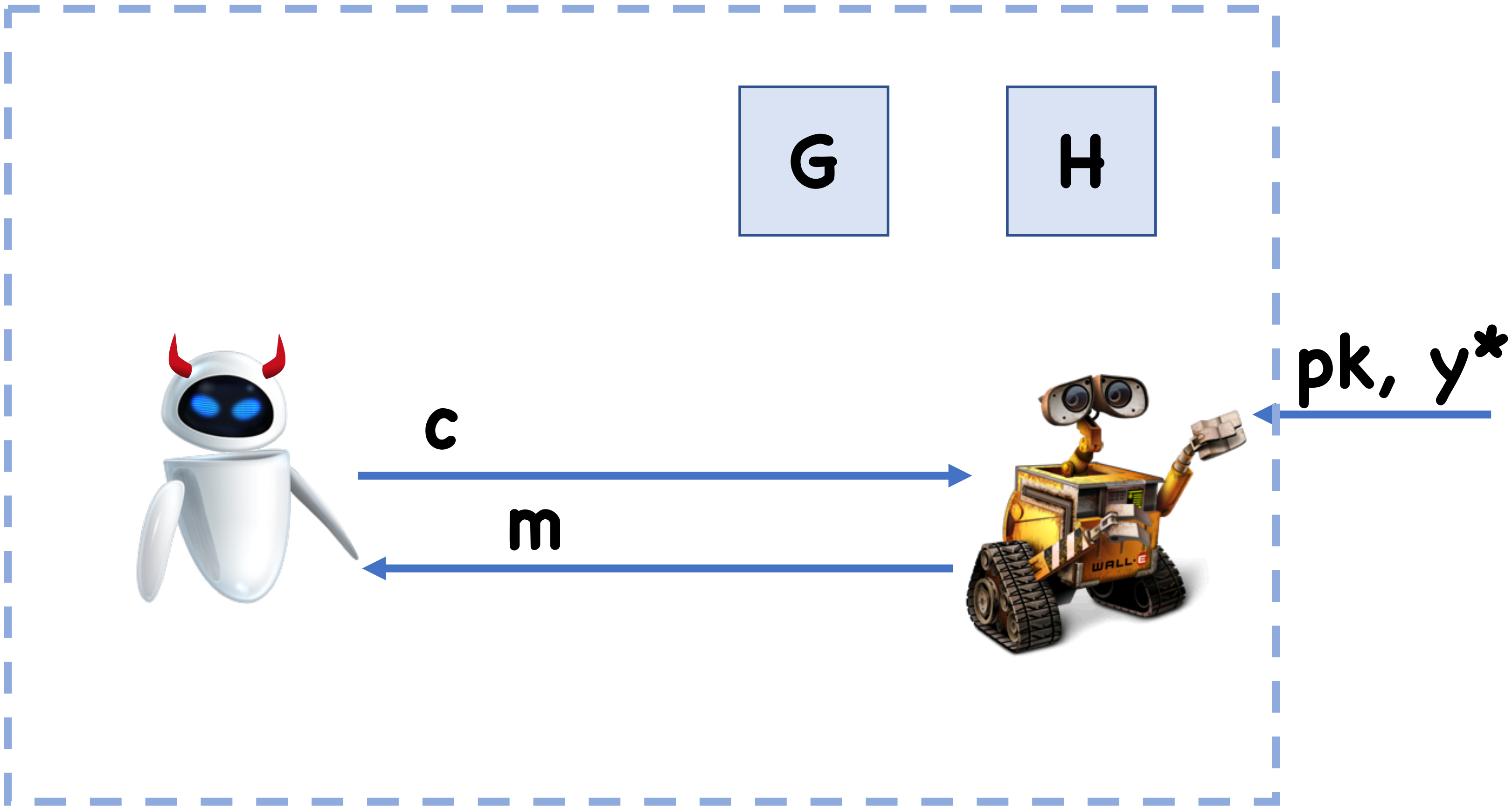
All ciphertexts decrypt to valid messages

- Makes it hard to argue security

High Level Proof Sketch



Claim: For any valid ctxt **c** queried by adv, adv must have created **c** by running **Enc(pk,m;r).** In this case, **m** can be decoded by looking at queries to **G,H**



Advantages of RSA-OAEP

RSA domain is at least 2048 bits

In hybrid encryption, ciphertext overhead =2048 bits

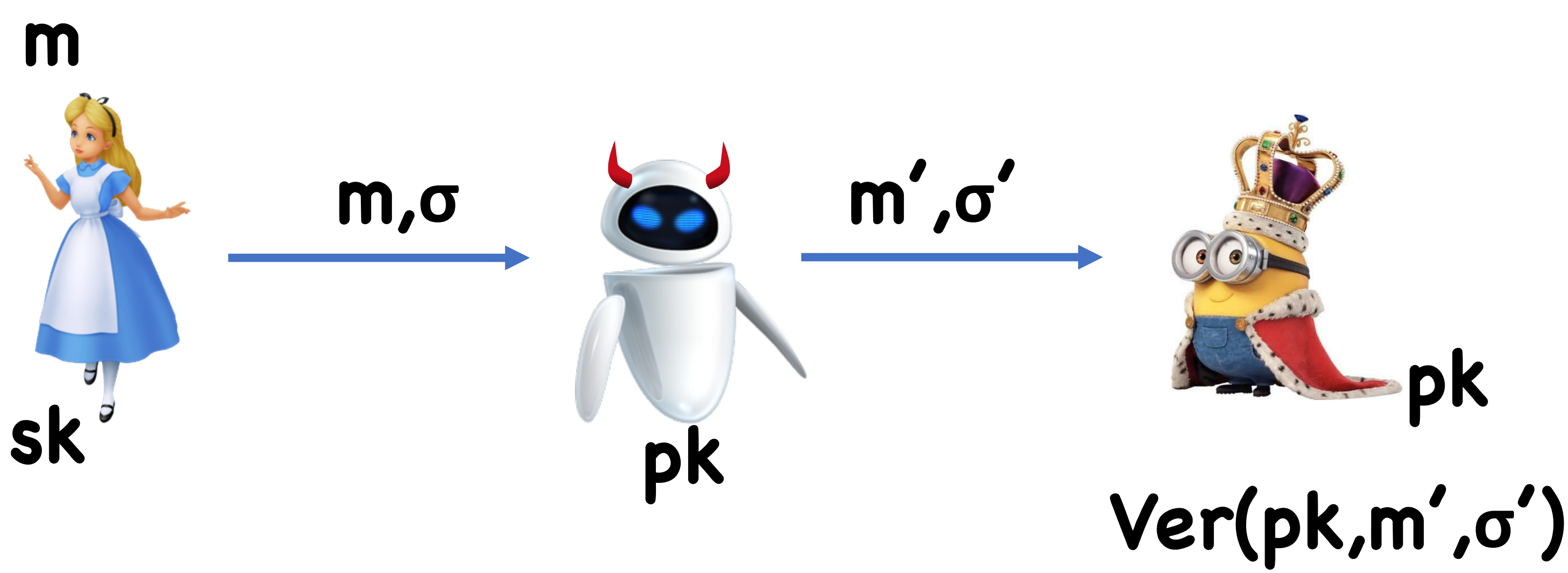
With OAEP (optimal asymmetric encryption padding), plaintext size can be, say 2048-256 bits with ciphertext size = 2048 bits

- Overhead = 256 bits

Digital Signatures

(aka public key MACs)

Message Integrity



Goal: If Eve changed **m**, Bob should reject

Syntax and Correctness

Algorithms:

- **Gen() → (sk,pk)**
- **Sign(sk,m) → σ**
- **Ver(pk,m,σ) → 0/1**

Correctness:

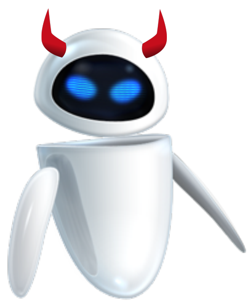
Pr[Ver(pk,m,Sign(sk,m))=1: (sk,pk)←Gen()] = 1

Security Notions?

Much the same as MACs, except adversary gets verification key

1-time Security For MACs

$(sk,pk) \leftarrow Gen()$

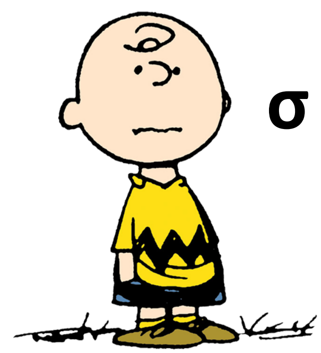


pk

m

σ

(m*,σ*)



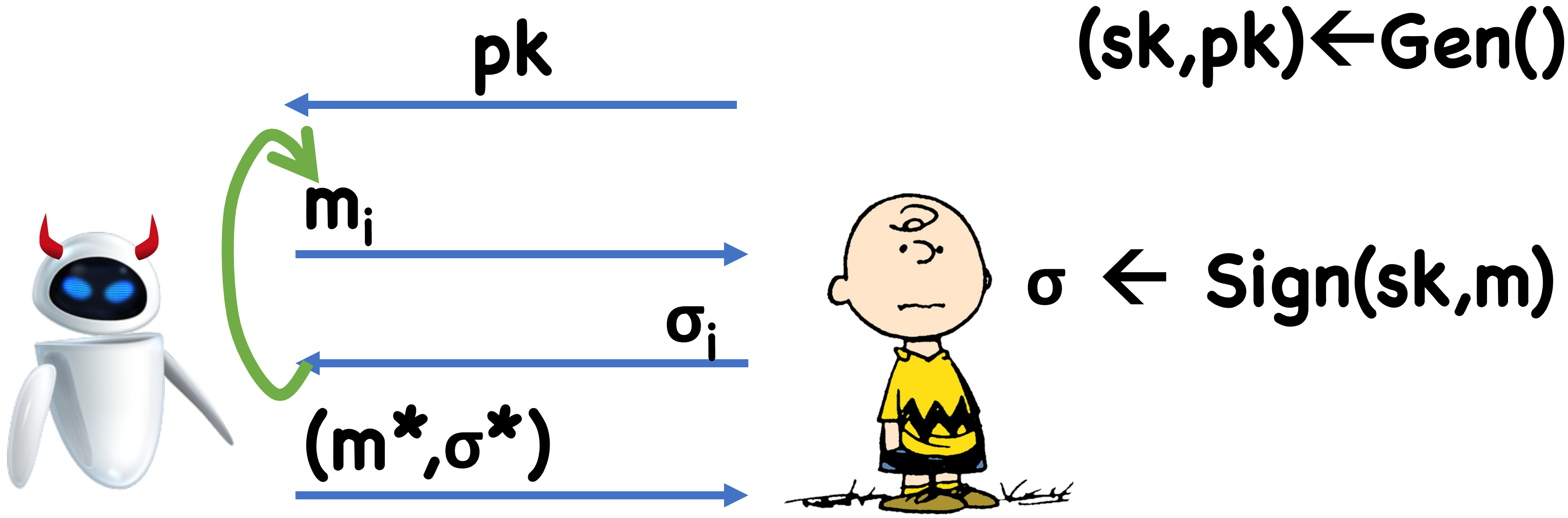
$\sigma \leftarrow Sign(sk,m)$

Output 1 iff:

- $m^* \neq m$
- $Ver(pk,m^*,\sigma^*) = 1$

1CMA-Adv() = Pr[ outputs 1]

Unbounded Use MACs



Output 1 iff:

- $m^* \notin \{m_1,\ldots\}$
- $\mathrm{Ver}(pk,m^*,\sigma^*) = 1$

CMA-Adv() = Pr[ outputs 1]

Signatures from TDPs?

$\mathrm{Gen}_{\mathrm{Sig}}() = \mathrm{Gen}()$

$\mathrm{Sign}(sk,m) = F^{-1}(sk,m)$

$\mathrm{Ver}(pk,m,\sigma)$: $F(pk, \sigma) == m$

Signatures from TDPs

$\mathbf{Gen_{Sig}() = Gen()}$

$\mathbf{Sign(sk,m) = F^{-1}(sk,\ H(m)\ )}$

$\mathbf{Ver(pk,m,\sigma):\ F(pk,\ \sigma) == H(m)}$

Theorem: If $\mathbf{(Gen,F,F^{-1})}$ is a secure TDP, and $\mathbf{H}$ is modeled as a random oracle, then $\mathbf{(Gen_{Sig},Sign,Ver)}$ is CMA-secure

Signatures from Injective TDFs?

What goes wrong?

Next Time

More digital signatures