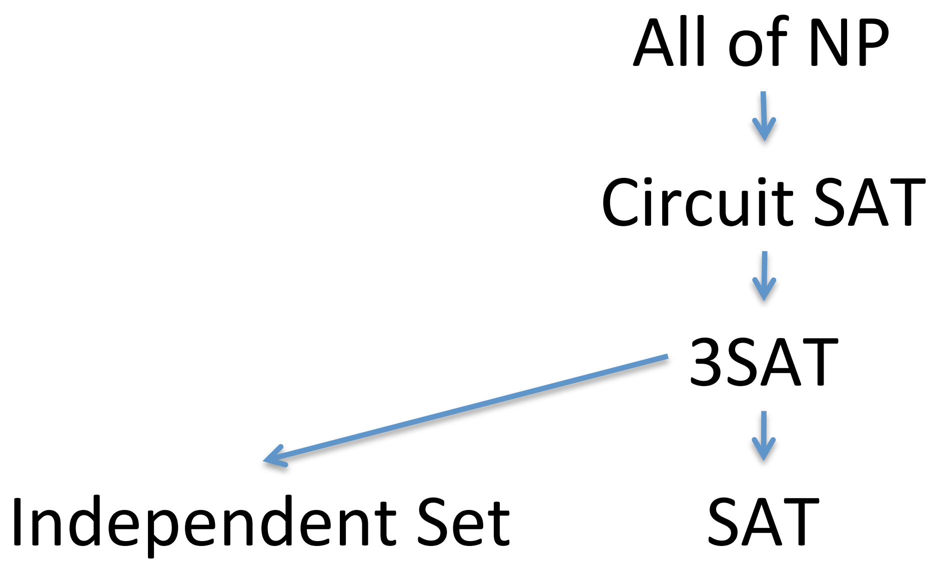
CS 161: Design and Analysis of Algorithms

NP-Complete II: More NP-Complete Problems

- The problems
- Some reductions

So Far



Recipe For Proving NP-Completeness

- Pick a well-known NP-Complete problem
- Prove that your problem can be used to solve the well-known NP-Complete problem
  - Given an instance of the well-known problem, construct an instance of your problem
  - Show that the instance of the well-known problem has a solution if and only if the instance of your problem does

Hamiltonian Cycle

- Given a graph $G = (V,E)$, is there a simple cycle that visits every node exactly once, and returns to the starting point?

Hamiltonian Path

- Given a graph G and two nodes s and t, is there a simple path from s to t that visits all nodes in G?

Longest Path

- Given a graph G and a goal b, determine if there is a simple path in G with length at least b

3D Matching

- n boys, n girls, n pets
- Set of triples (b,g,p) that means b, g, and p go well together
- Find a way to match up each boy, girl, and pet

Vertex Cover

- Find a collection of at most b nodes that touch every edge in the graph

Set Cover

- Given a set E and a collection of subsets $\{S_i\}$, pick at most b subsets such that their union is E

Clique

- A **clique** of a graph G = (V,E) is a set of n nodes such that every two nodes in the set have an edge between them
- The Clique problem is to, given a goal g, find a clique on at least g nodes

Subset Sum

- Given a set of integers v_i, find a subset of the integers whose sum is exactly V

Knapsack

- Given a set of of items {1,…,n}, weights for each item w_i, value of each item v_i, weight capacity W, and a target value V, find a subset of the items whose total weight is at most W and whose value is at least V

Integer Linear Programming

$$\max \sum_i c_i x_i$$

$$\sum_i A_{j,i} x_i \leq b_j \forall j$$

$$x_i \geq 0 \forall i$$

$$x_i \in \mathbb{Z}$$

Zero-One Equations

$$\text{find } x_i$$

$$\sum_i A_{j,i} x_i = 1 \forall j$$

$$A_{j,i} \in \{0,1\}$$

$$x_i \in \{0,1\}$$

Scheduling

- Given a set of n jobs, can only work on 1 at a time
- Job i is available to start working on at time r_i, due by time d_i, and has duration t_i
- Can we complete all the jobs before their deadlines?

The Reductions

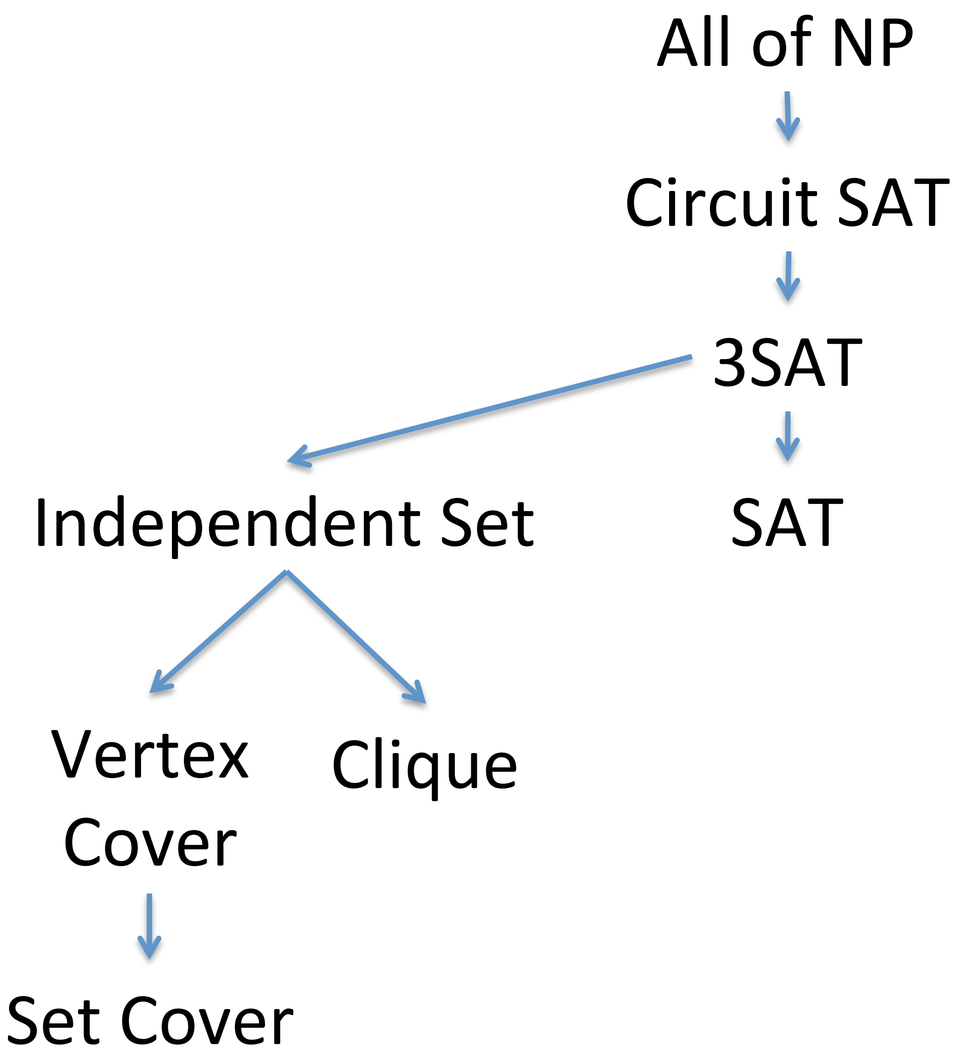
Independent Set $\leq_P$ Vertex Cover

- S is an independent set if and only if V-S is a vertex cover
- Given a graph G and a goal b, for the independent set problem, construct (G,|V|-b) as an instance of Vertex Cover

Independent Set $\leq_P$ Clique

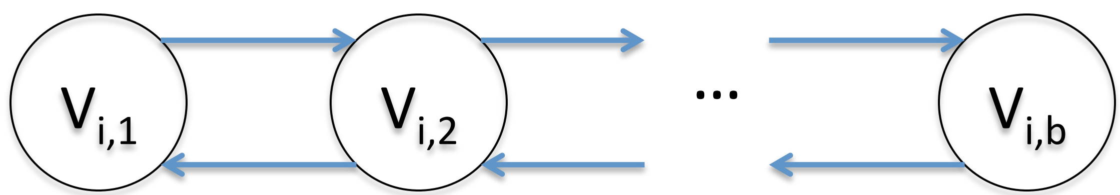
- Define the complement G* = (V,E*) of a graph G = (V,E) where E* consists of every edge not in E
- S is an independent set of G if and only if S is a clique in G*
- Given a graph G and a goal b for the Independent Set problem, simply compute (G*,b) as an instance of Clique

So Far


All of NP
Circuit SAT
3SAT
SAT
Independent Set
Vertex Cover
Clique
Set Cover

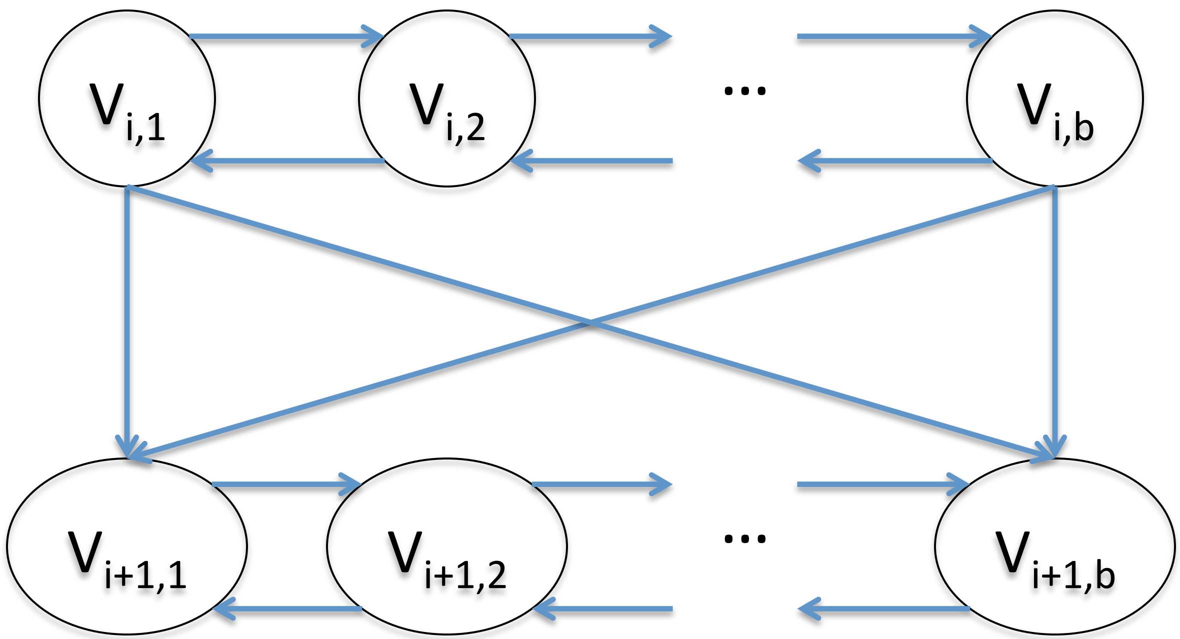
3SAT $\leq_P$ Directed Hamiltonian Cycle

- Given a 3SAT instance with n variables, k clauses
- Pick some b >> k (to be determined later)
- Construct n paths P_i



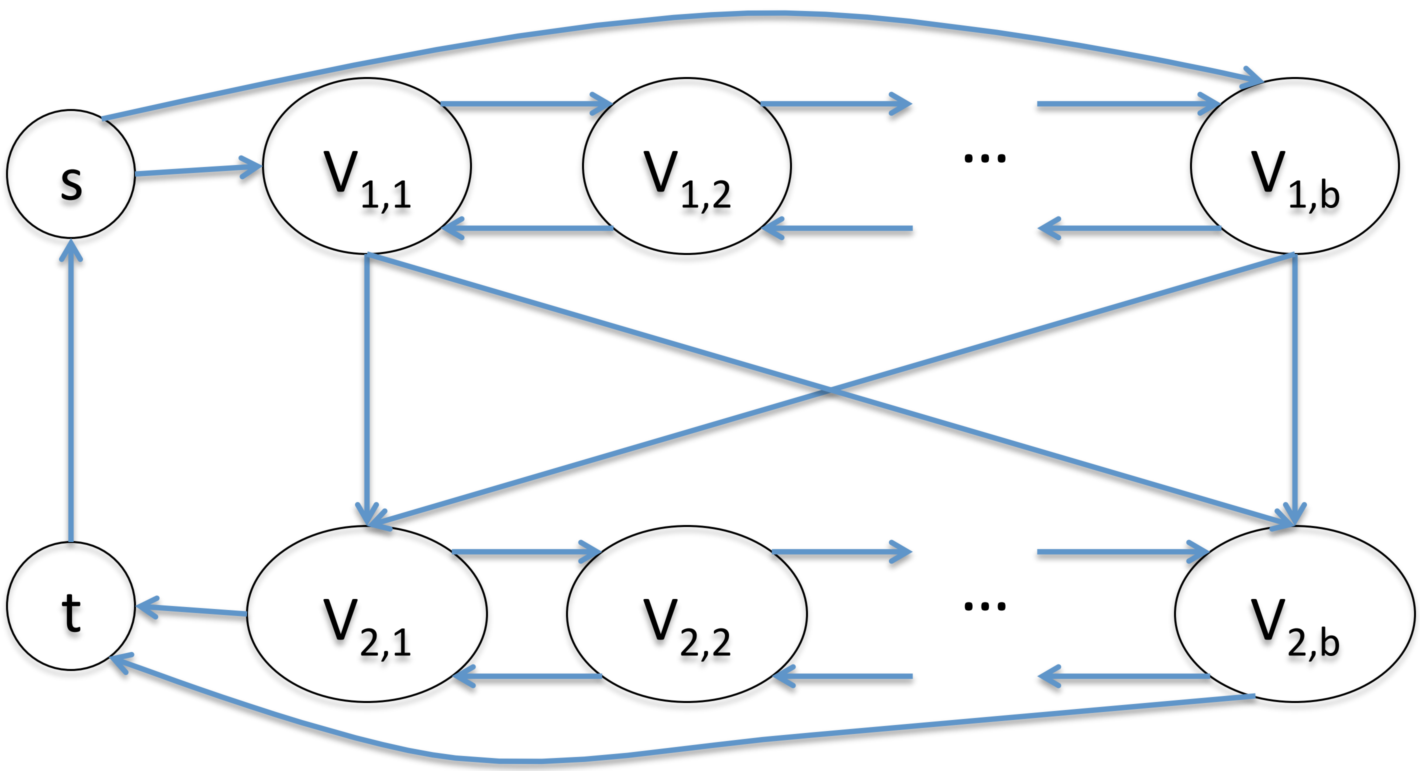
3SAT $\leq_P$ Directed Hamiltonian Cycle

- Add edges $(v_{i,1}, v_{i+1,1})$, $(v_{i,1}, v_{i+1,b})$, $(v_{i,b}, v_{i+1,1})$, $(v_{i,b}, v_{i+1,b})$



3SAT $\leq_P$ Directed Hamiltonian Cycle

- Add a source node s, target node t, and 5 edges: $(s,v_{1,1})$, $(s,v_{1,b})$, $(v_{n,1},t)$, $(v_{n,b},t)$, (t,s)

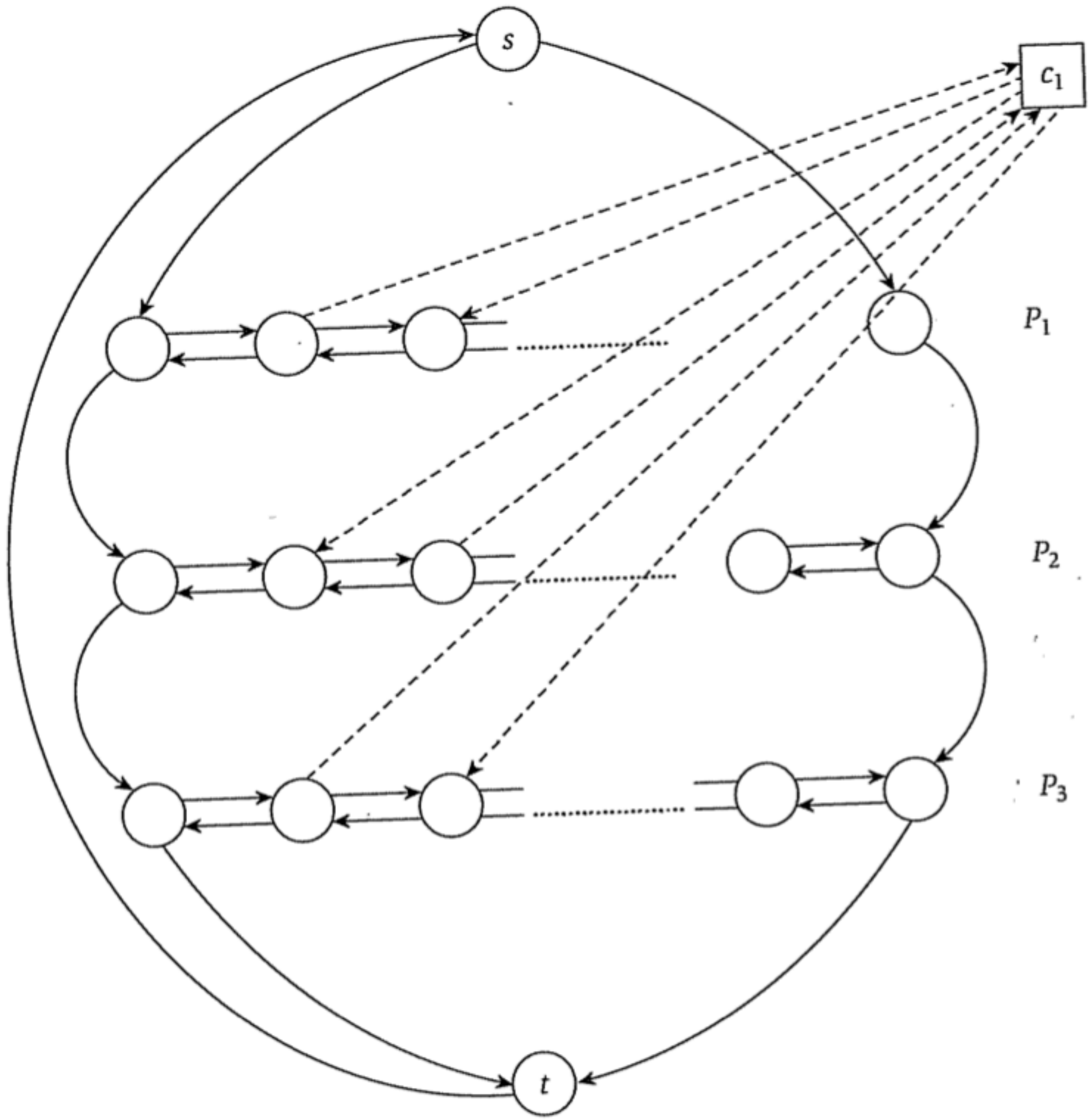


3SAT $\leq_P$ Directed Hamiltonian Cycle

- Any cycle through this graph must hit every path P_i and travel through it
- For each path, have a choice: travel from right to left or from left to right
- 2^n different cycles
- Identify with variable assignments: if P_i goes left-to-right, x_i true, false otherwise

3SAT $\leq_P$ Directed Hamiltonian Cycle

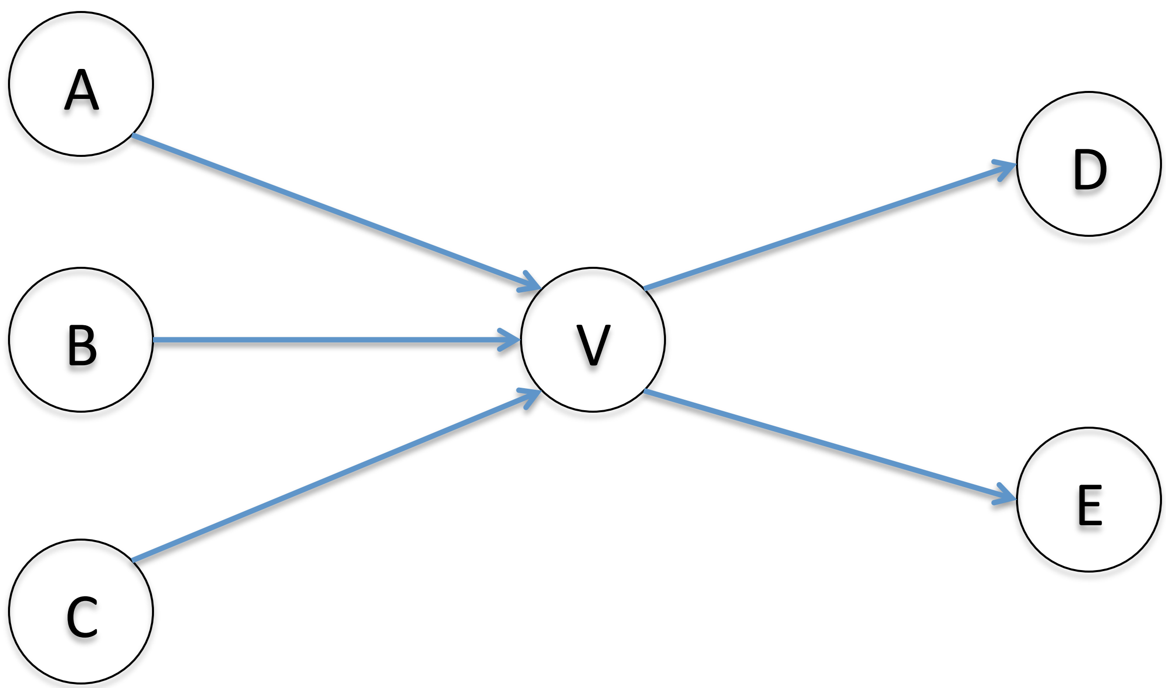
- How to enforce clauses?
- Example: $(x_1 \vee \overline{x_2} \vee x_3)$
  - Either path P_1 goes left to right, or P_2 goes right to left, or P_3 goes left to right
  - How do we enforce this?


s
c_1
P_1
P_2
P_3
t

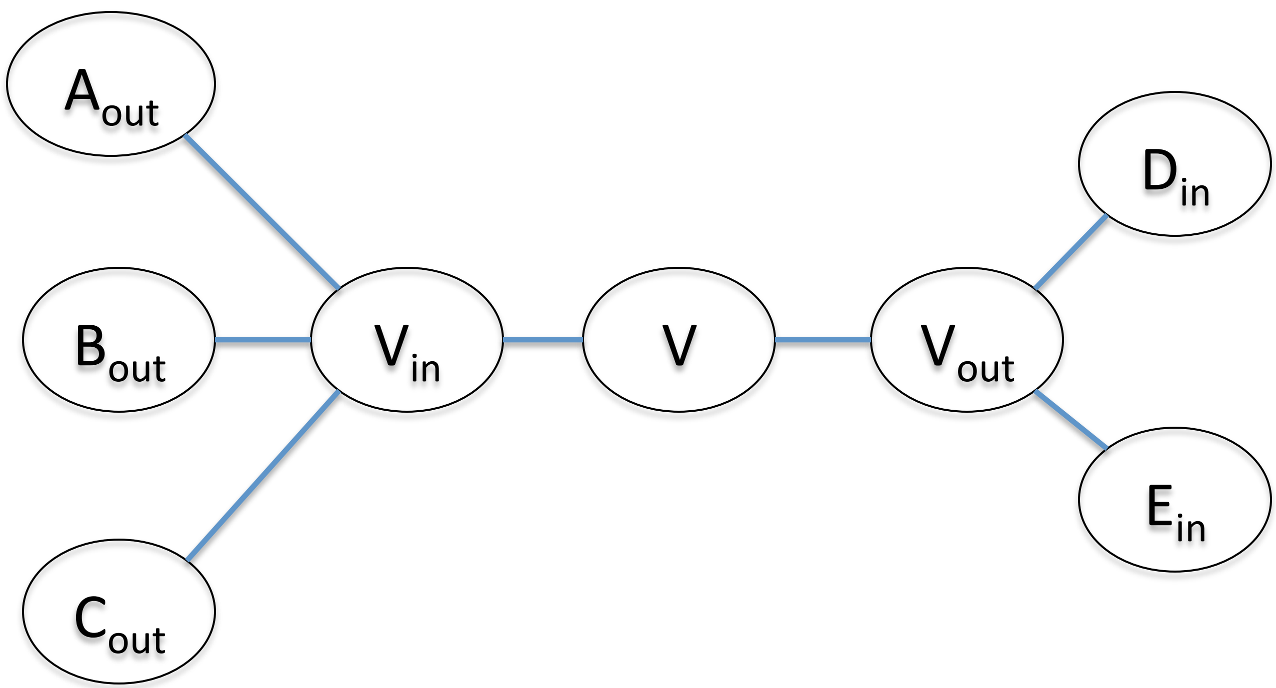
3SAT $\leq_P$ Directed Hamiltonian Cycle

- If our 3SAT instance has a satisfying assignment, there will be a Hamiltonian path in G where each path P_i is traveled in the direction indicated by x_i
- If G has a Hamiltonian path, then we can get a satisfying assignment by setting x_i to the direction P_i was traversed
- Note: delete edge (t,s), and we can show that Directed Hamiltonian Path is also NP-Complete

Directed Hamiltonian Cycle $\leq_P$ Hamiltonian Cycle



Directed Hamiltonian Cycle $\leq_P$ Hamiltonian Cycle



Directed Hamiltonian Cycle $\leq_P$ Hamiltonian Cycle

- If G has a directed Hamiltonian Cycle, G has an undirected Hamiltonian Cycle
  - If we follow edge (u,v) in G, we follow edges (u,u_{out}), (u_{out},v_{in}), (v_{in},v)

Directed Hamiltonian Cycle $\leq_P$ Hamiltonian Cycle

- If G' has an undirected Hamiltonian Cycle, pick some node v, and orient the cycle so that the path goes from v_{in} to v to v_{out}
- Next node after v_{out} has to be w_{in} for some other node w
- Must visit w next (otherwise, we will never be able to visit w in the future)
- Then must visit w_{out}

Directed Hamiltonian Cycle $\leq_P$ Hamiltonian Cycle

- Thus, we always visit nodes in order u_{in}, u, u_{out}.
- To construct Hamiltonian path in G: If we follow some edge (u_{out}, u'_{in}) in G', follow the edge (u,u') in G
- Therefore, G has a directed Hamiltonian cycle if and only if G' has an undirected Hamiltonian cycle
- Note: reduction also words for Directed Hamiltonian Path $\leq_P$ Hamiltonian Path

Hamiltonian Cycle $\leq_P$ TSP

- Given a graph G = (V,E), construct a new weighted complete graph G' on V as follows:
  - If e is in E, the w(e) = 1
  - If e is not in E, then w(e) = 1 + x
    - Determine x later

Hamiltonian Cycle $\leq_P$ TSP

- If G has a Hamiltonian cycle, then that cycle is a tour of length |V| in G'
- Similarly, if G' has a tour of length |V|, all edges must be of weight 1, they must be present in G. Therefore, we have a Hamiltonian cycle in G

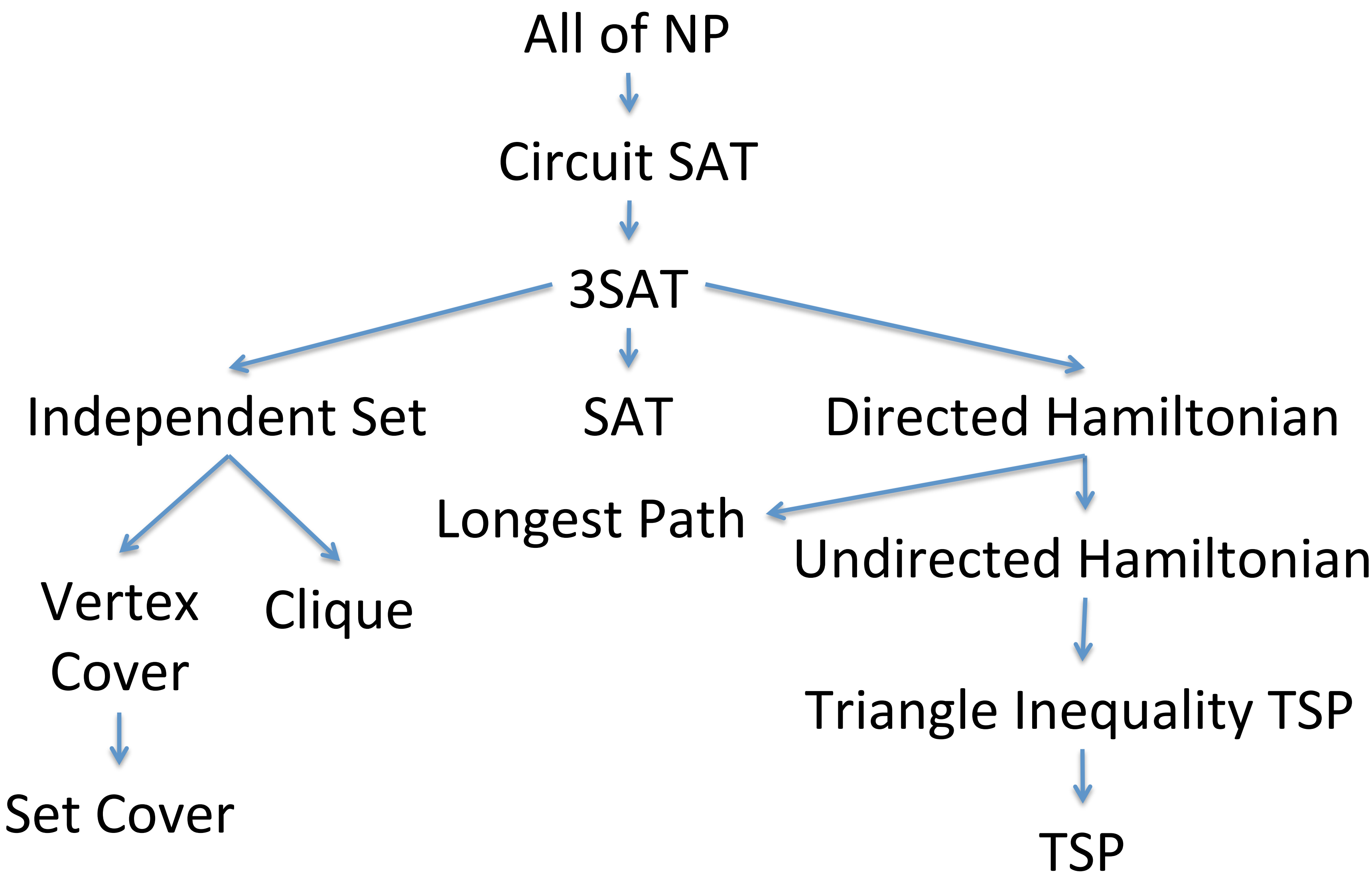
$x = 1$

- All edges have weight 1 or 2
- Triangle inequality satisfied:
  - $w(u,v) + w(v,w) \geq w(u,w)$
- Therefore, TSP where edge weights satisfy triangle inequality is still NP-Complete

Large x

- Notice in general that either there is a tour of length n, or the lightest tour has length at least n + x
- What if we could approximate TSP to within a factor of p(n) for some polynomial n
- Set x = n p(n)
- If there is a Hamiltonian cycle, then there is a cycle with cost n → obtain cycle with cost ≤ n p(n) → must have optimal cycle

So Far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Triangle Inequality TSP
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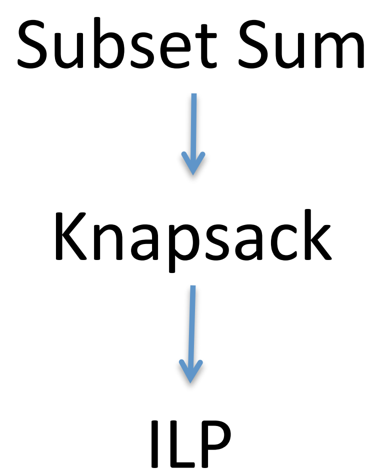
TSP

Problems We Haven't Proved

- 3D Matching
- ILP
- ZOE
- Subset Sum
- Knapsack
- Scheduling



Coping With NP-Completeness

Coping With NP-Completeness

- Suppose we need to solve some *optimaztion* problem
- We realize that the decision version is NP-Complete
- We probably cannot solve all instances exactly
- Approximate?

Approximation Algorithms

- Recall our greedy algorithm for set cover
  - Repeatedly pick the set that contains the most number of uncovered elements
  - If there is a set cover of size k, greedy returns a set cover of size at most k ln n
- Approximation ratio: ratio of obtained solution to optimal solution
  - Set cover: ln n

Vertex Cover

- Special case of set cover, so also a ln n ratio
- Can we do better?

Vertex Cover

- Vertex Cover: set of nodes touching every edge
- Matching: Set of edges that don’t share endpoints
- Any vertex cover is at least as large as any matching

Vertex Cover

- Maximal Matching: a matching such that we can't add any more edges
  - Easy to compute: keep adding an edge until we can't any more
- Algorithm: compute maximal matching M, and return the set S containing both endpoints of every edge of M
  - $|S| = 2|M|$

Vertex Cover

- Is S a vertex cover?
  - Say some edge e has both endpoints not in S
  - Then we can add e to the matching M, meaning M wasn't maximal
- Let optimal vertex cover have size Opt
- $|S| = 2|M| \leq 2$ Opt
- Approximation ratio = 2

Knapsack

- For any constant ε, possible to devise a polynomial time algorithm with an approximation ratio of 1+ε
- Called a polynomial time approximation scheme (PTAS)

Travelling Salesman

- We briefly saw that Triangle Inequality TSP can be approximated within a factor of 2
- General TSP: cannot be approximated to within any polynomial unless P = NP
  – Reduction from Hamiltonian

Not all NP-Complete Problems are Created Equal

- Polynomial time approximation scheme
  - Subset sum, Euclidean TSP
- Constant factor approximation
  - Vertex cover, Triangle Inequality TSP
- Some polynomial factor
  - Set cover
- No Approximation
  - TSP