CS 161: Design and Analysis of Algorithms

Linear Programming II: Duality/Reductions

- Recap
- Example
- Duality
- Reductions

Recap

Objective Function

$$\max \sum_i c_i x_i$$

$$\sum_i A_{j,i} x_i \leq b_j \forall j$$

Constraints

$$x_i \geq 0 \forall i$$

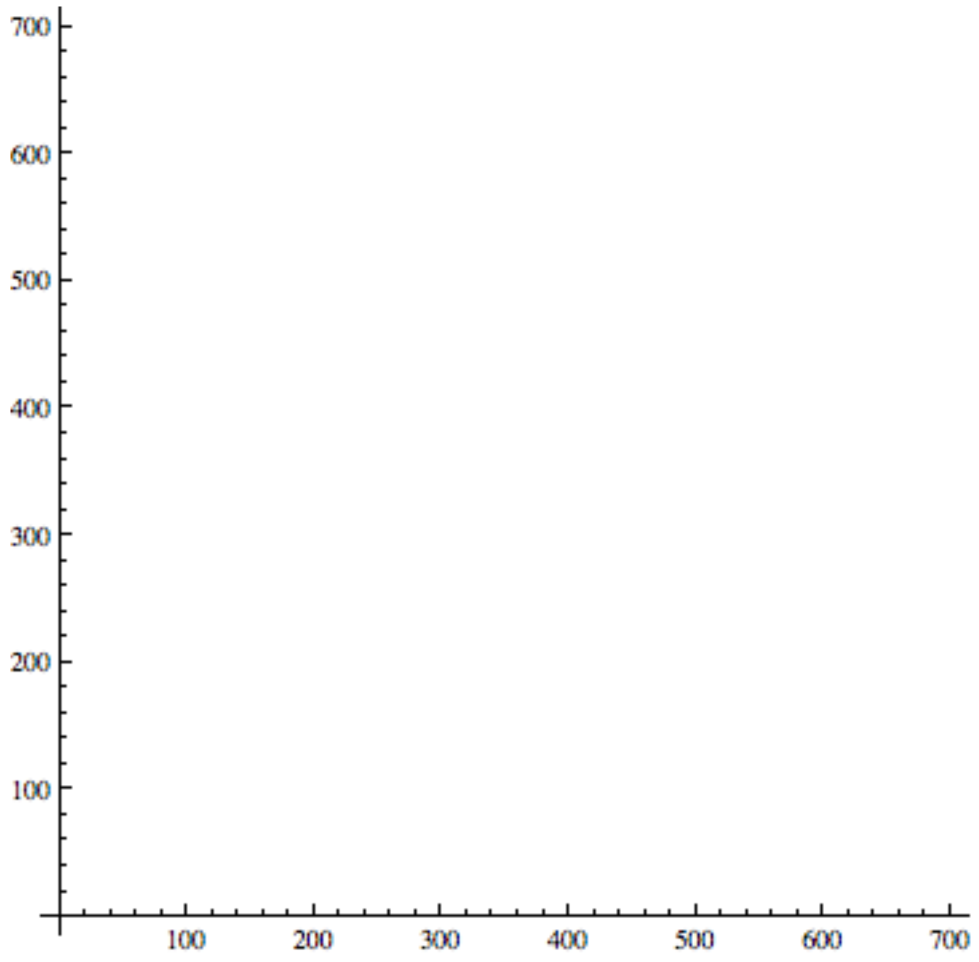
Profit Maximization

- Suppose a candy company can make two types of candy
- The company can produce up to 500 boxes a day of the first type, each box making the company $5
- They can produce up to 300 boxes of the second type, which box making them $10
- The company can only produce 600 boxes of candy per day

Profit Maximization

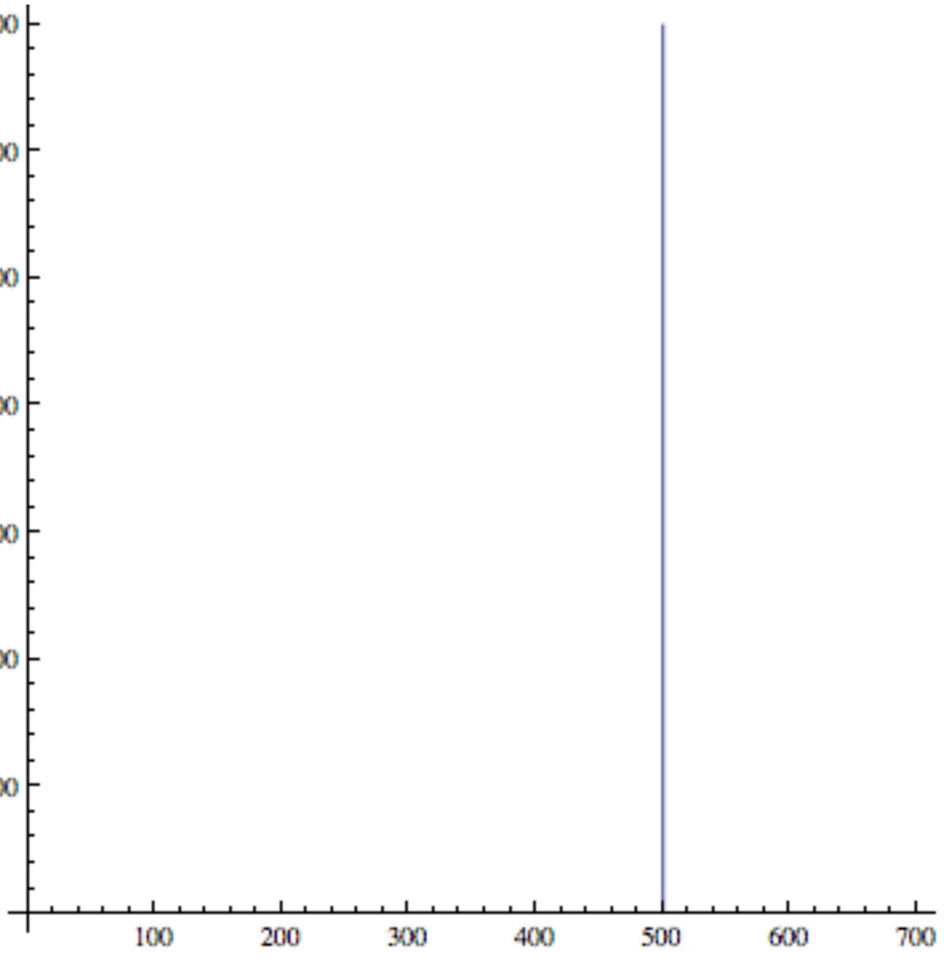
- Variables x_1 and x_2 represent the number of boxes of candy 1 and 2 produced
- $0 \leq x_1, x_2$
- $x_1 \leq 500$
- $x_2 \leq 300$
- $x_1 + x_2 \leq 600$
- Maximize: $5\ x_1 + 10\ x_2$

Simplex Method



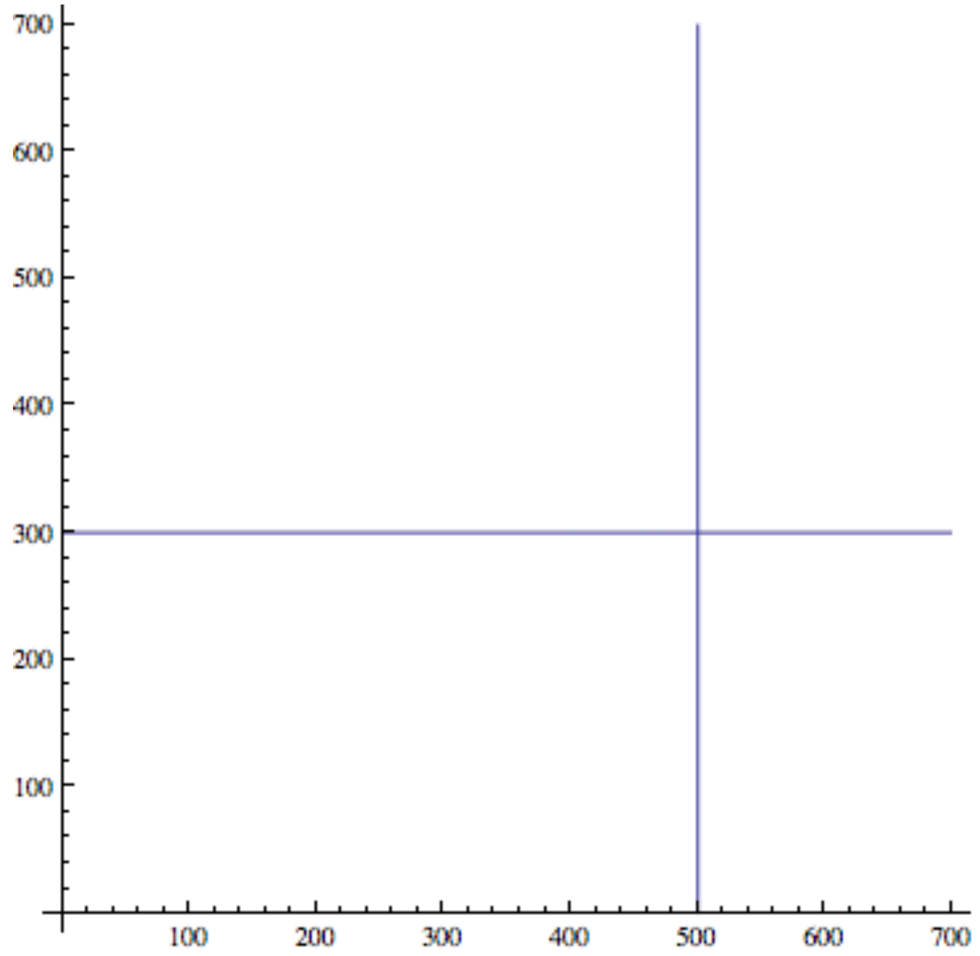
Simplex Method

- $x_1 \leq 500$



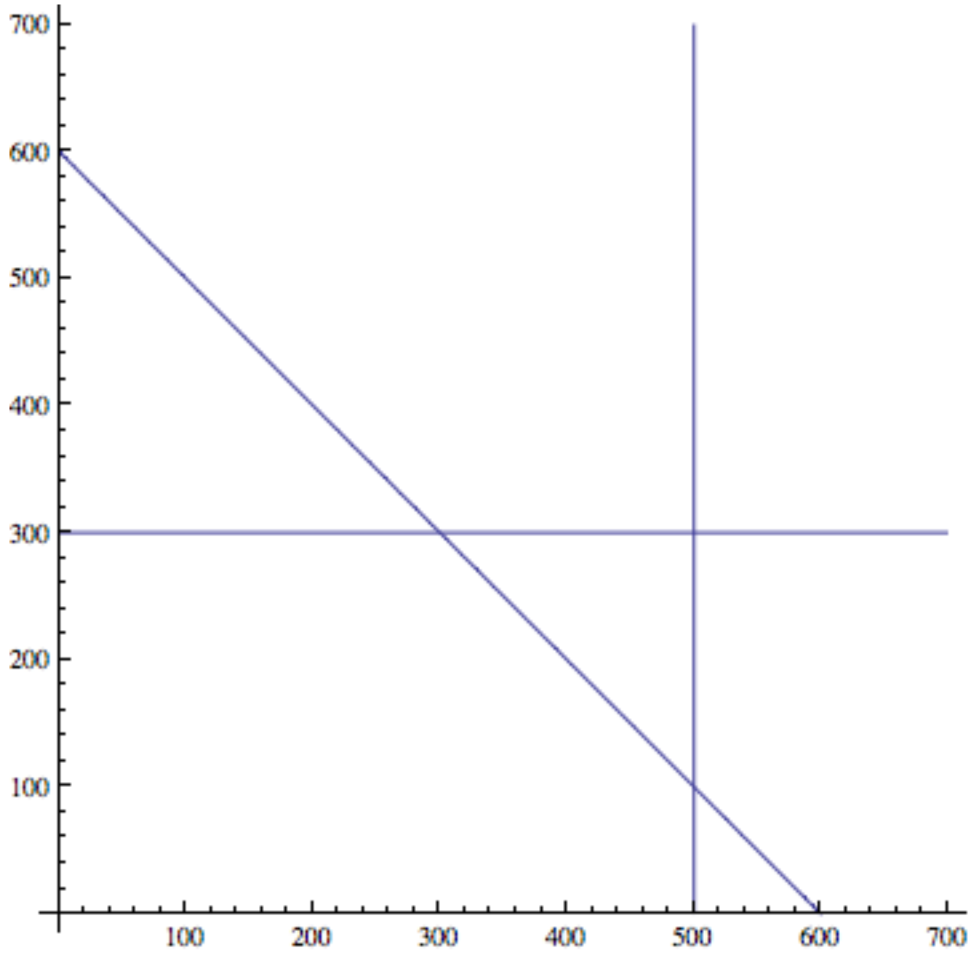
Simplex Method

- $x_2 \leq 300$

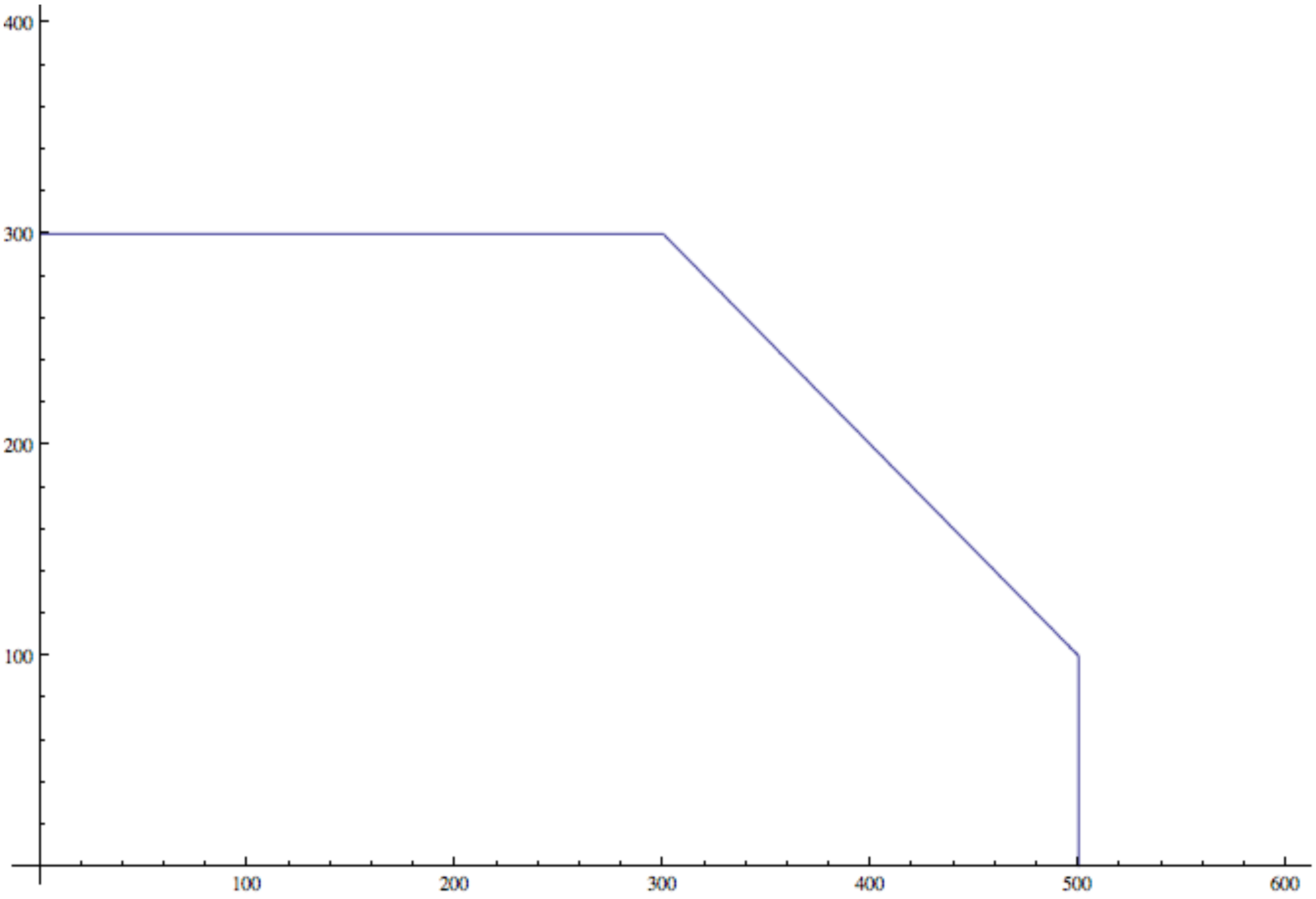


Simplex Method

- $x_1 + x_2 \leq 600$



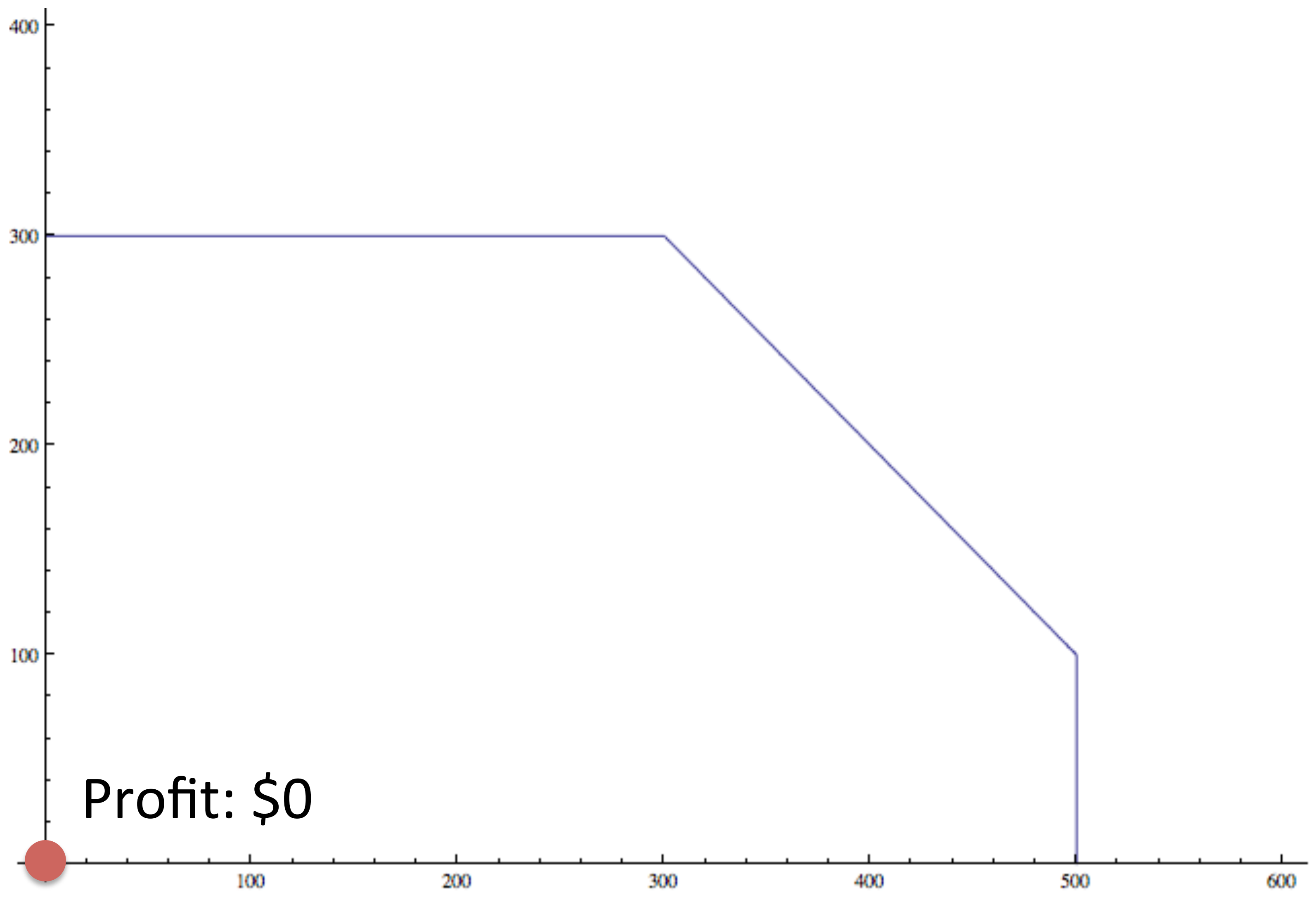
Feasible Region



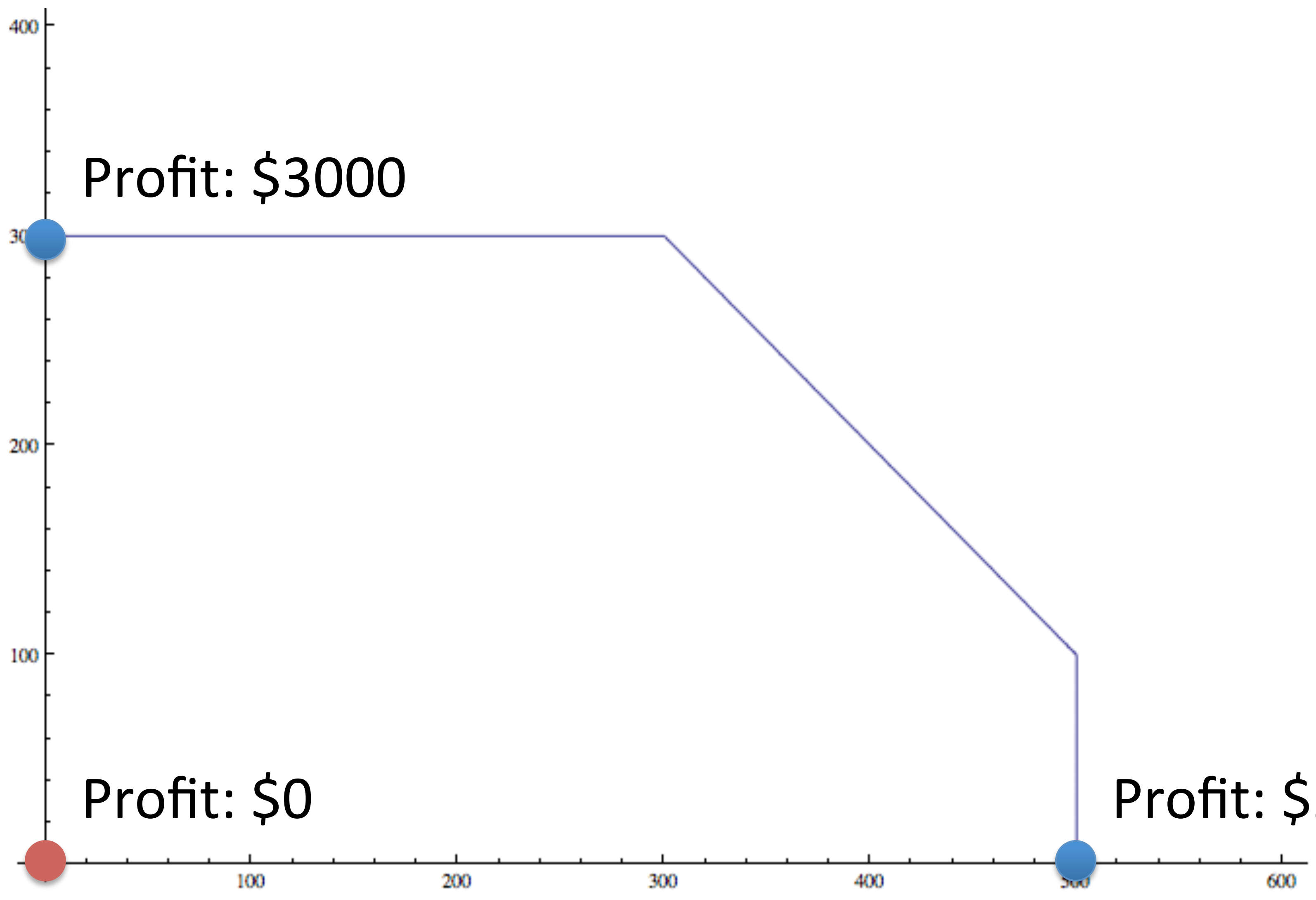
Simplex Method

- Recall:
  - Start at any vertex of the feasible region
  - Repeatedly move to neighboring vertex with more optimal solution

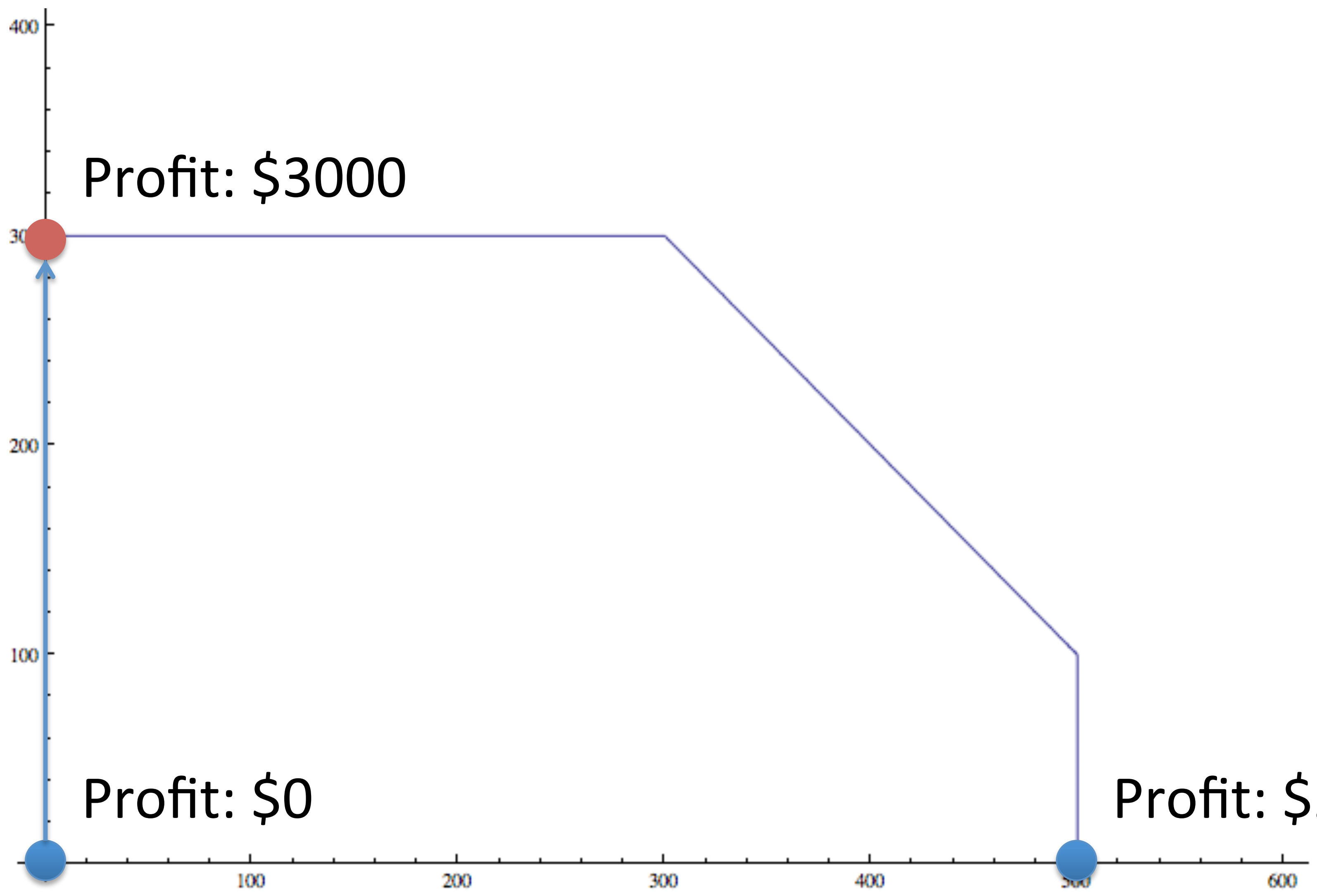
Simplex Method



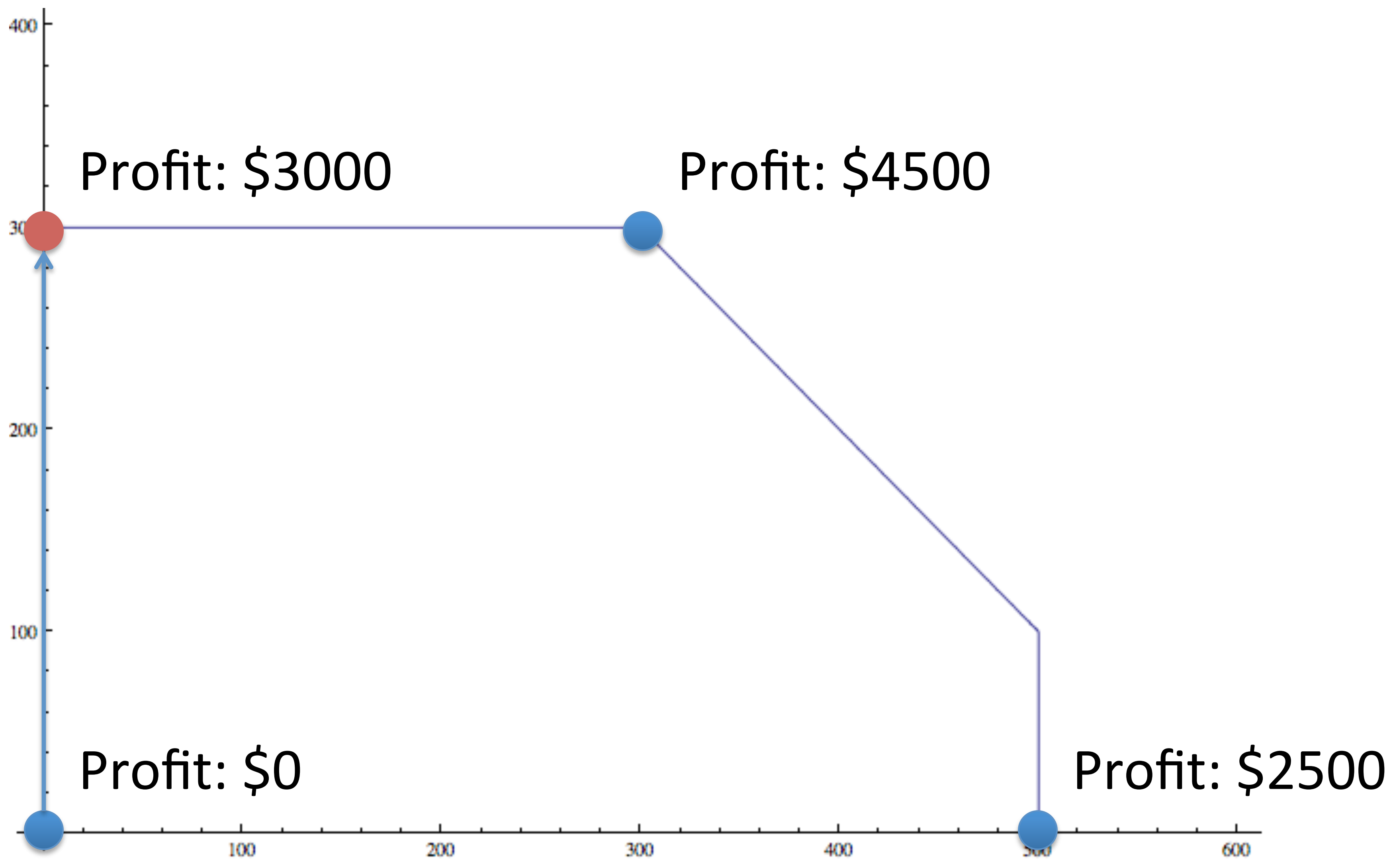
Simplex Method



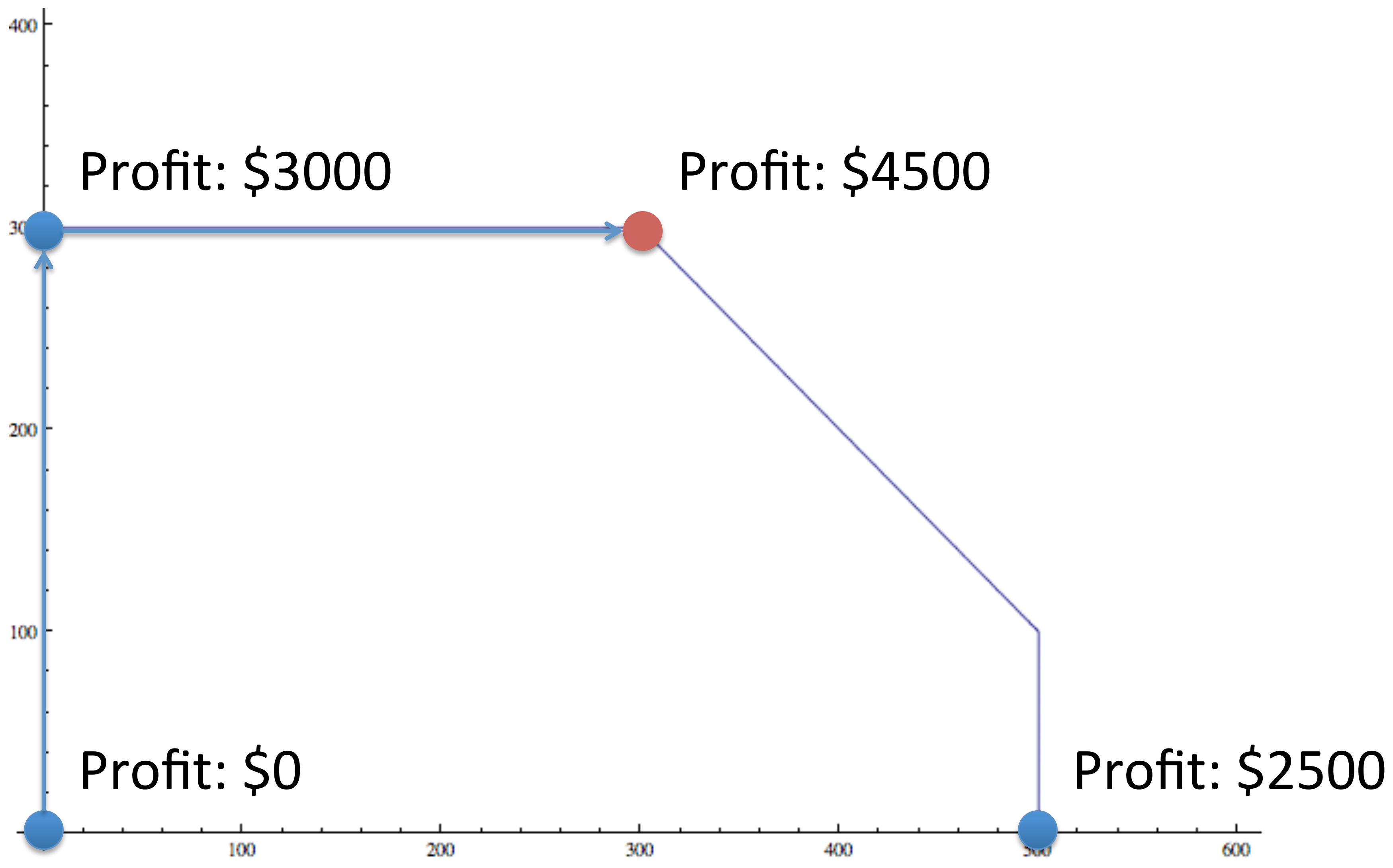
Simplex Method



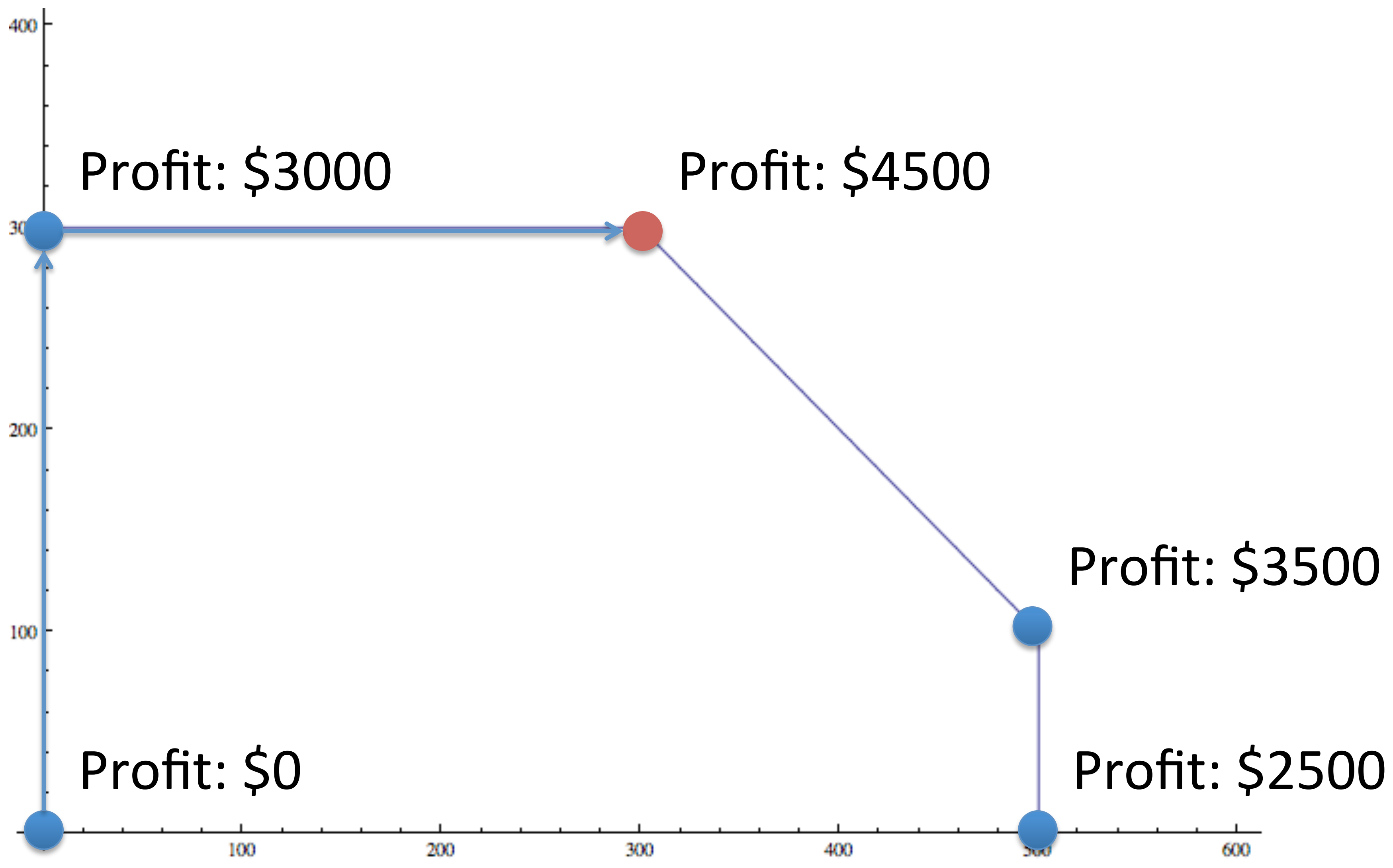
Simplex Method



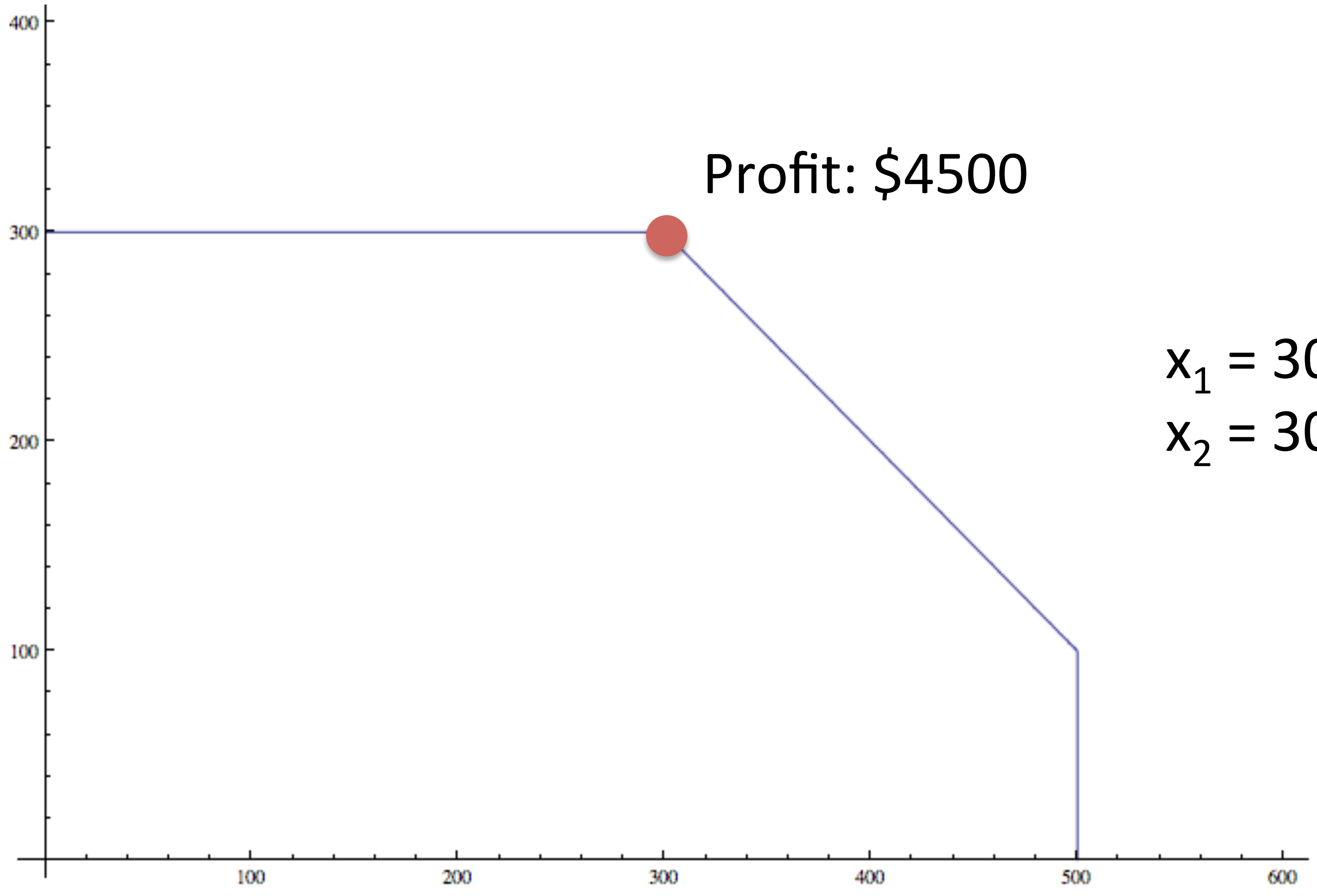
Simplex Method



Simplex Method



Simplex Method

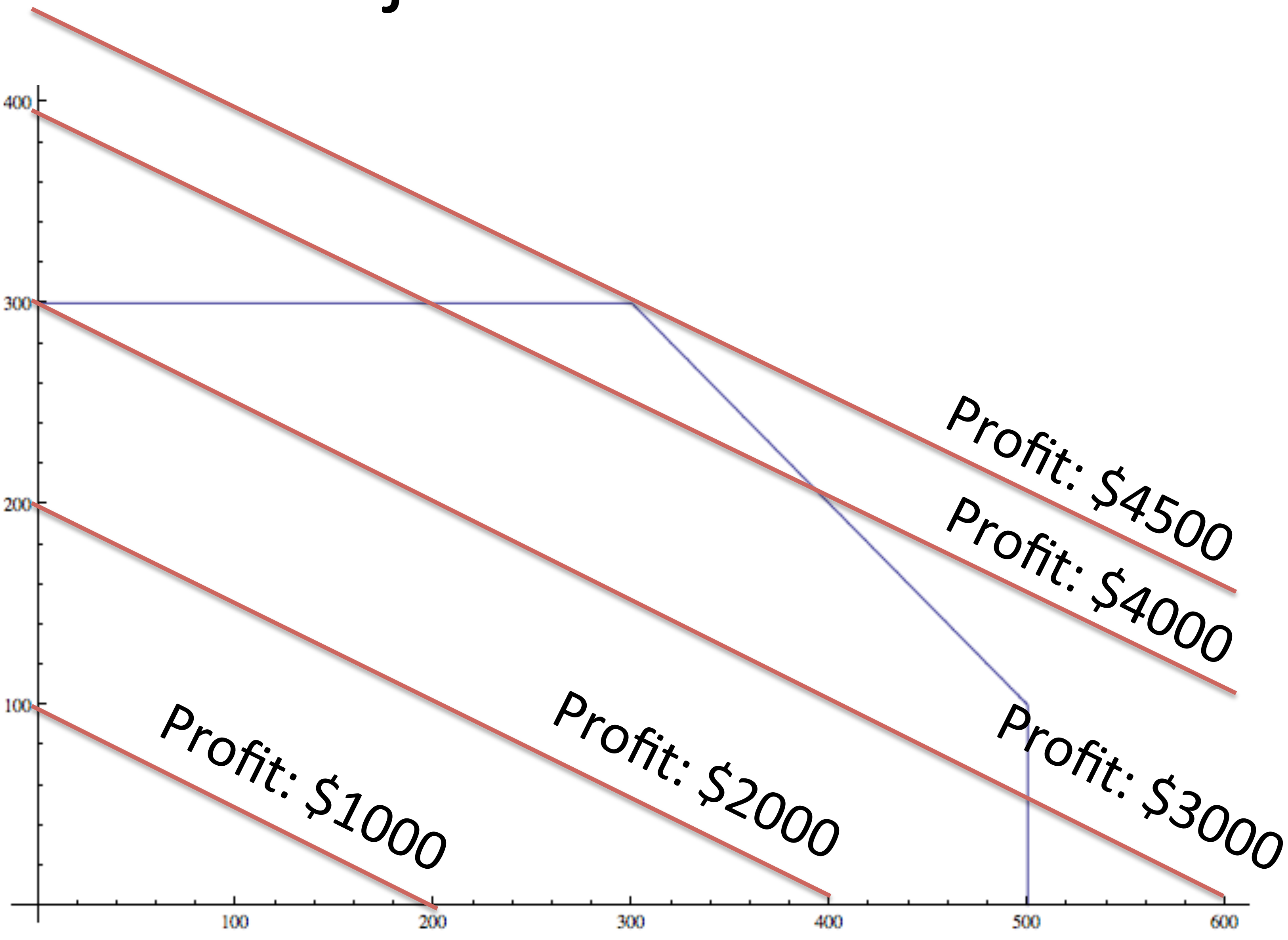


$x_1 = 300$
$x_2 = 300$

Why is a vertex always optimal?

- Set of points where objective function is equal to a constant c forms a hyperplane
- Change value of objective function by shifting this hyperplane
- Hyperplane must intersect feasible region
- Objective function maximized when intersection is at an extreme

Objective Function



Proof of Optimality?

- We wish to prove that our solution is optimal be showing that there is no way to make more than $4500

Proof of Optimality?

- Recall the LP:
  - Maximize: $5 x_1 + 10 x_2$ subject to the constraints
  - $0 \leq x_1, x_2$
  - $x_1 \leq 500$
  - $x_2 \leq 300$
  - $x_1 + x_2 \leq 600$

Proof of Optimality

- Let's try combining constraints
- $x_1 + x_2 \leq 600 \rightarrow 5x_1 + 5x_2 \leq 3000$
- $x_2 \leq 300 \rightarrow 5x_2 \leq 1500$
- Add together: $5x_1 + 10x_2 \leq 4500$
- But $5x_1 + 10x_2$ is just the objective function!
- Therefore, the objective function is always at most \$4500, so our solution is optimal

Proof of Optimality

- Goal: combine constraints together to get a bound on the objective function

$$\max \sum_i c_i x_i$$

$$\sum_i A_{j,i} x_i \leq b_j \forall j$$

$$x_i \geq 0 \forall i$$

Proof of Optimality

$$\sum_j y_j \left(\sum_i A_{j,i} x_i \right) \leq \sum_j b_j y_j$$

Proof of Optimality

- Want bound on objective function, so want

$$\sum_i c_i x_i \leq \sum_i \left(\sum_j A_{j,i} y_j \right) x_i$$

Proof of Optimality

- Sufficient condition:

$$c_i \leq \sum_j A_{j,i} y_j$$

Proof of Optimality

- Thus, as long as

$$c_i \leq \sum_j A_{j,i} y_j$$

- We have that

$$\sum_i c_i x_i \leq \sum_j b_j y_j$$

Proof of Optimality

- Goal:

$$\min \sum_j b_j y_j$$

$$\sum_j A_{j,i} y_i \geq c_i \forall i$$

$$y_i \geq 0 \forall i$$

Duality

- Our goal then is to solve another linear program!
- This alternate linear program is known as the **dual** of the original program
- By construction, optimal solution of dual is at least optimal solution of primal
- **Duality Theorem**: optimums coincide

Matrix Notation

$$\max \mathbf{c}^T \mathbf{x} \qquad \min \mathbf{y}^T \mathbf{b}$$

$$\mathbf{A}\mathbf{x} \leq \mathbf{b} \qquad \mathbf{y}^T \mathbf{A} \geq \mathbf{c}^T$$

$$\mathbf{x} \geq 0 \qquad \mathbf{y} \geq 0$$

Reductions

- We already saw that linear programming can be used to solve the max flow problem
- What we showed was a **reduction**: Given an instance of the max flow problem, we:
  - Construct a linear program
  - Solve the linear program
  - Convert solution of linear program into solution for max flow

Reduction

- In general, solving problem A reduces to solving problem B if:
  - Given an instance of problem A, we can efficiently compute an instance of problem B***
  - Given a solution to the instance of problem B, we can efficiently construct a solution to the instance of problem A***

*** Need to define “efficient”

The Power of Reductions

- If solving problem A reduces to solving problem B, then we can reuse an algorithm to solve problem B in order to solve A
  - Convert instance of A into instance of B
  - Solve B using our algorithm
  - Convert solution to solution for A

The Power of Reductions

- What makes linear programming so powerful is that many problems can be reduced to linear programs

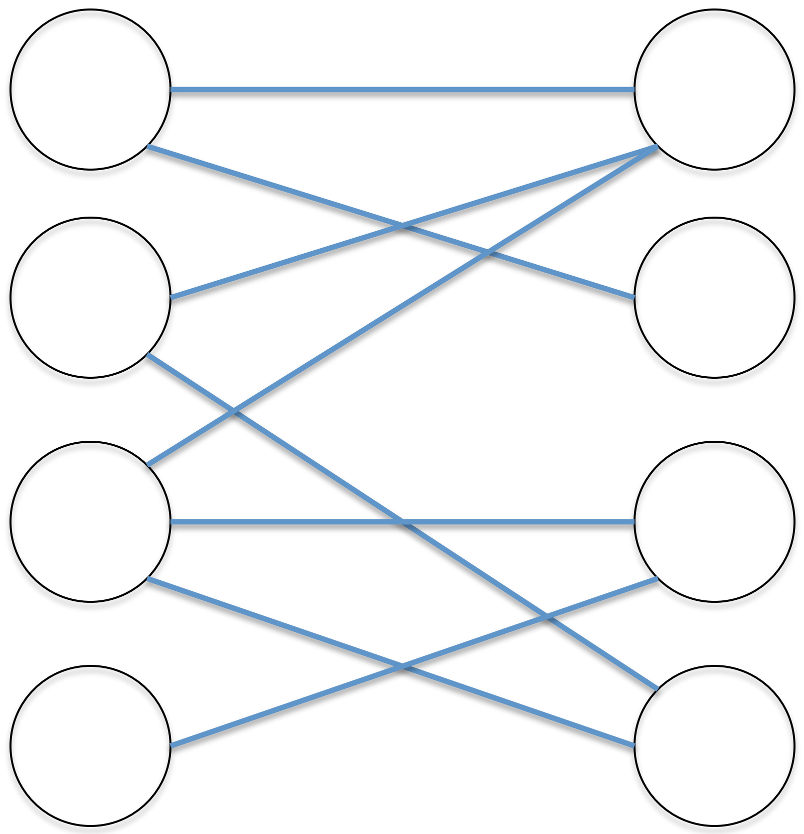
Reductions so Far

- Max Flow
- Profit Maximization

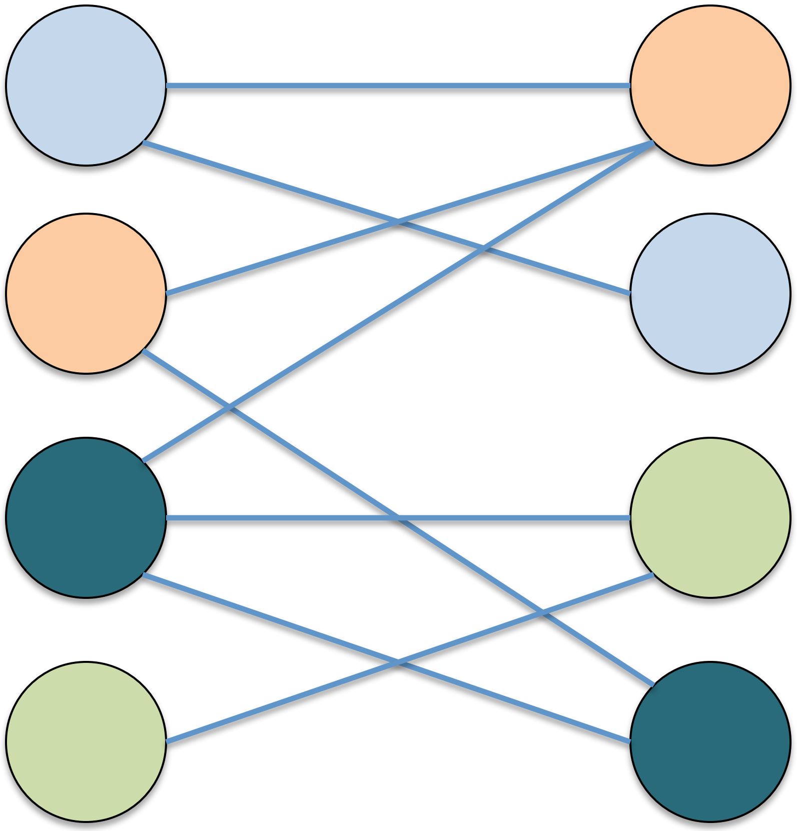
Bipartite Matching

- Suppose we have a list of n boys and n girls, and set of pairs (i,j) that mean boy i and girl j like each other
- Can we pair every boy with a girl so that each pair likes each other?

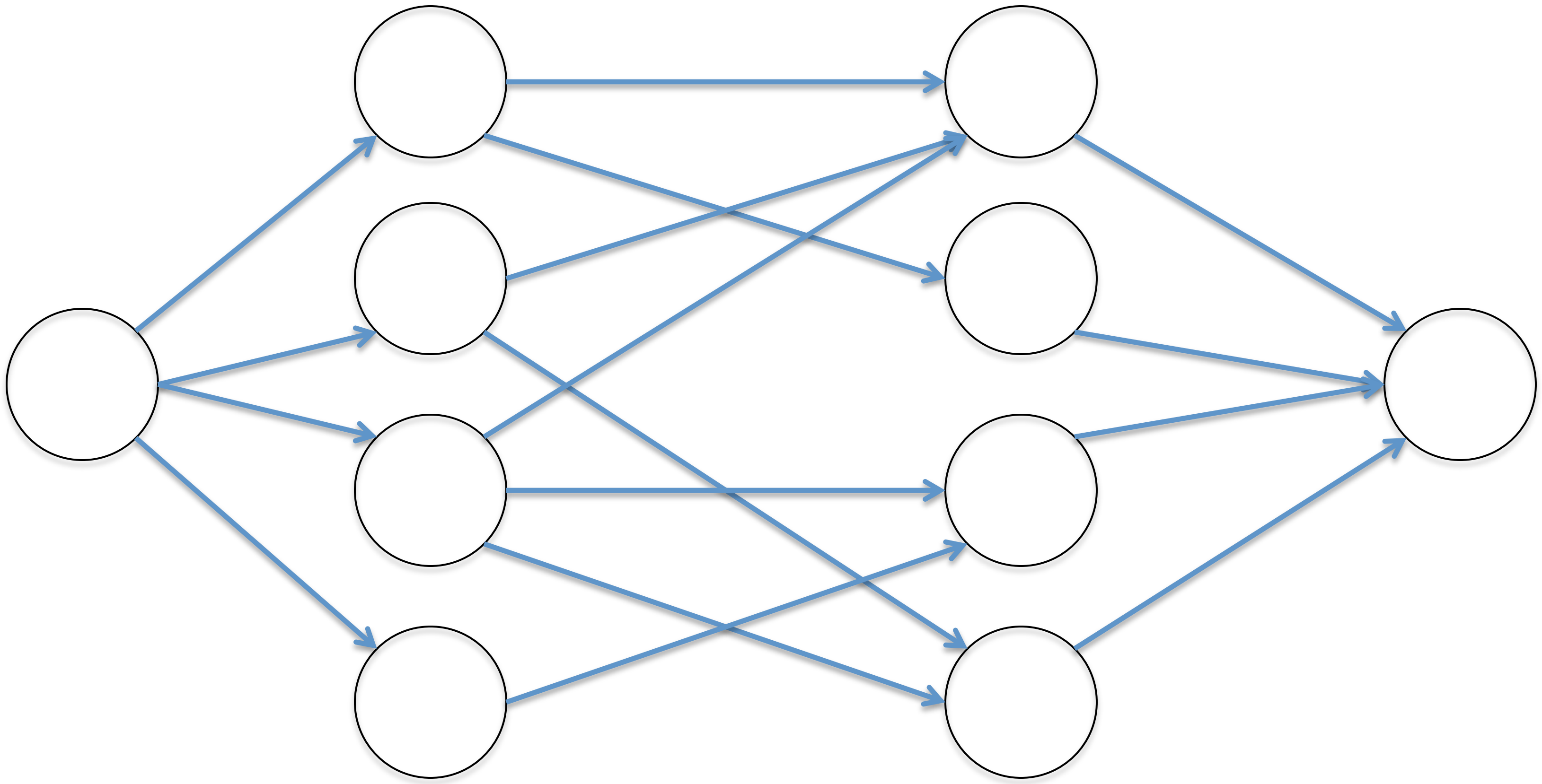
Bipartite Matching



Bipartite Matching



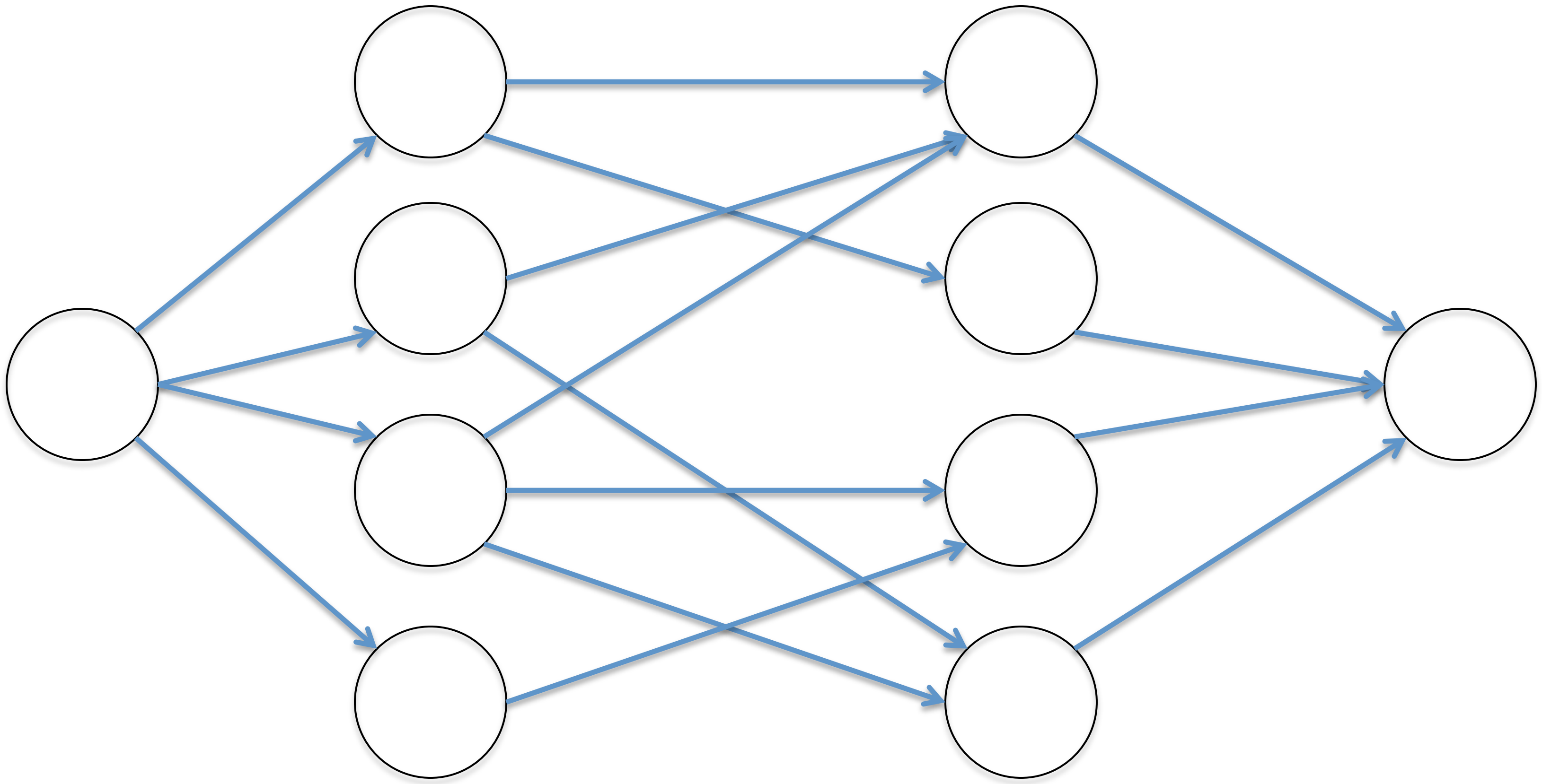
Bipartite Matching as Maximum Flow



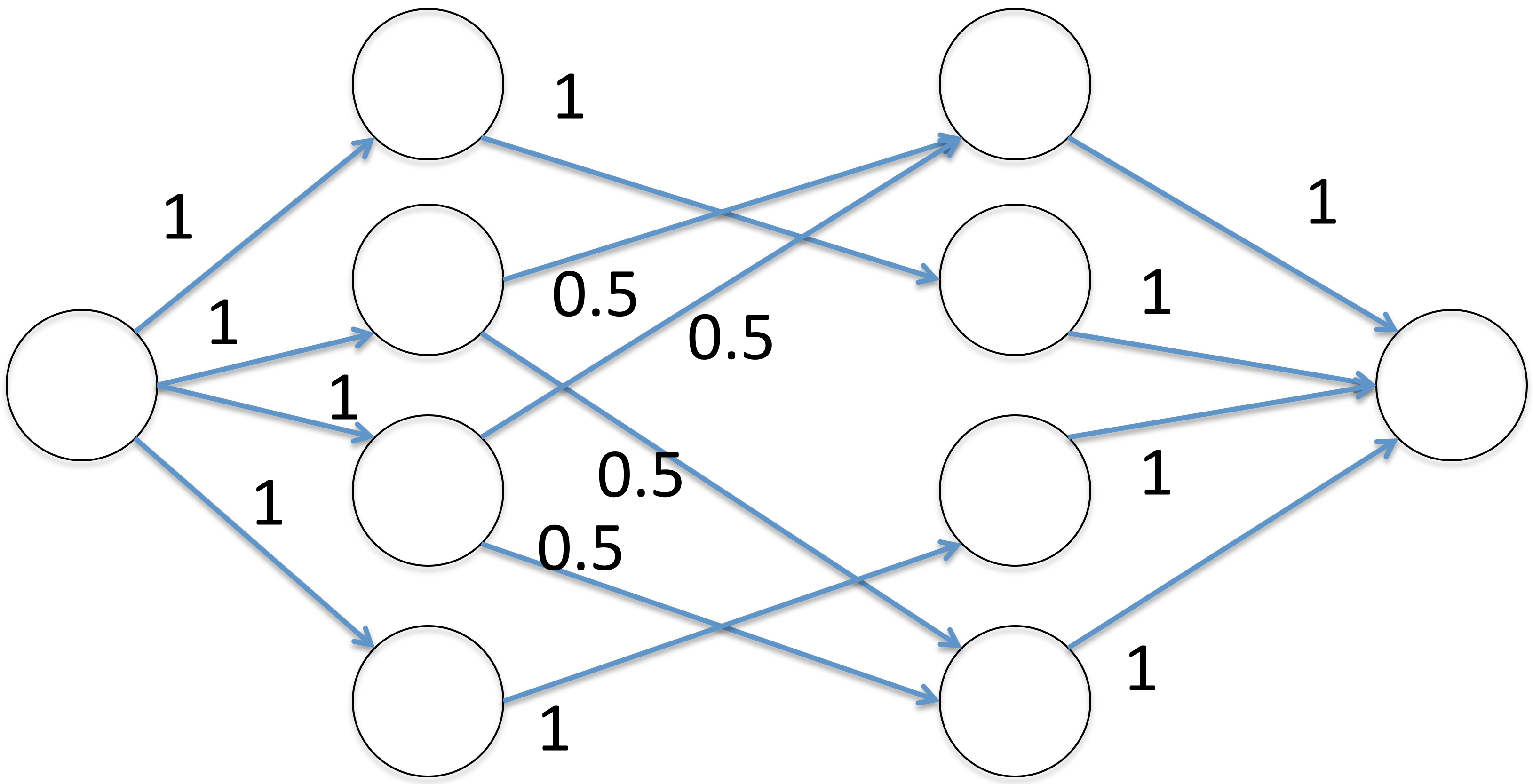
Bipartite Matching as Maximum Flow

- To see if there is a perfect matching, direct all edges from boy to girl
- Add a source node s with edges to each boy
- Add a sink node t with edges from each girl
- Compute the max flow
- If there is a flow equal to n, then there is a perfect matching

Bipartite Matching as Maximum Flow



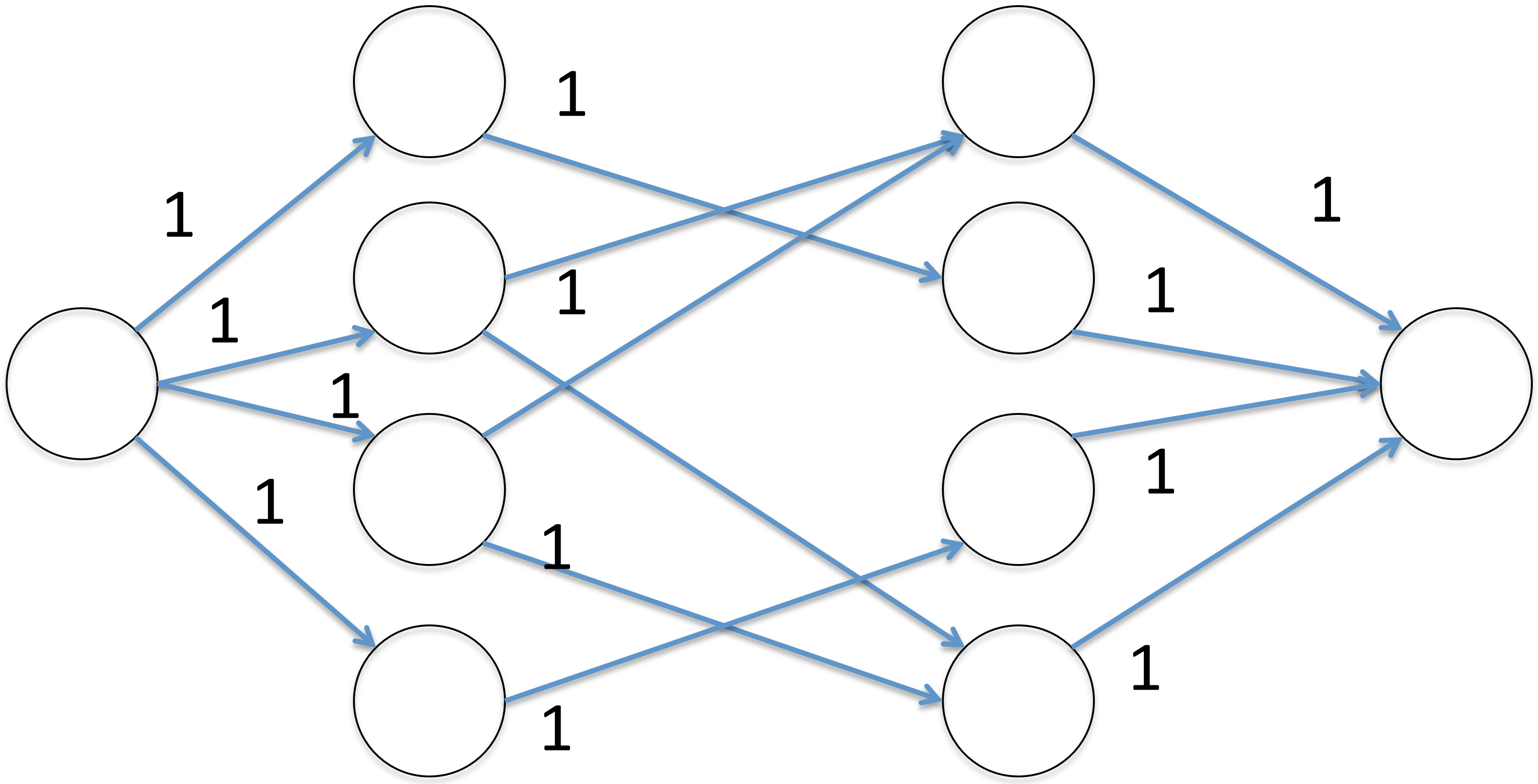
Problem?



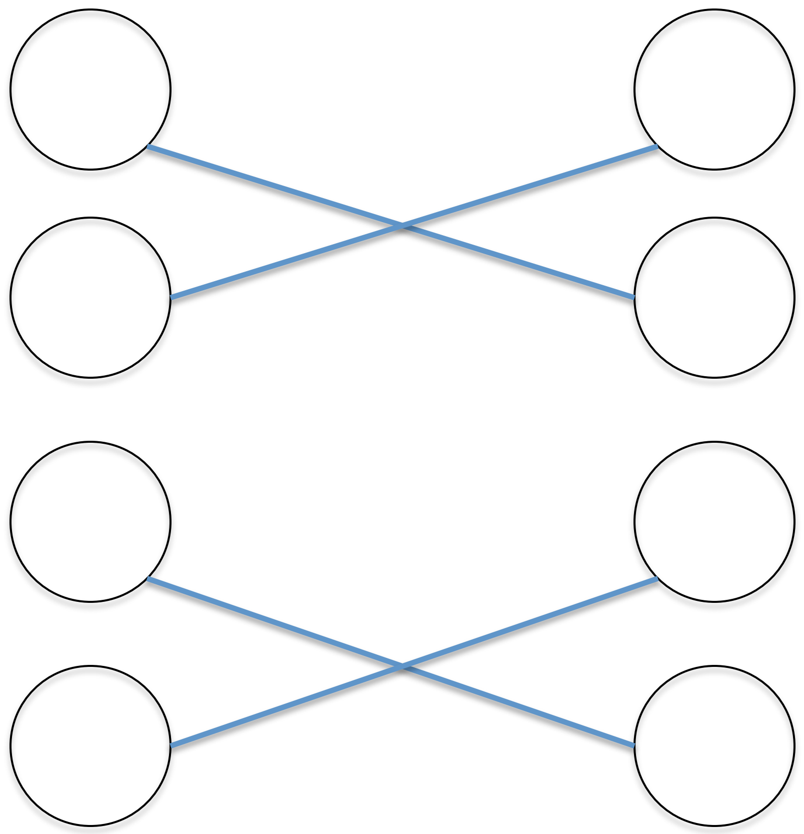
Bipartite Matching as Maximum Flow

- Our Max Flow algorithm always produces an integer flow if the edge weights are integers
  - Always increments flow by integer value
- Therefore, the maximum flow in the bipartite matching problem has integer flow on each edge
- To get matching, take edges with flow =1

Bipartite Matching as Maximum Flow



Bipartite Matching as Maximum Flow



Linear Programming for Shortest Path

- Can think of a path from s to t as a flow of size 1 from s to t
- Shortest path problem: find flow that minimizes total weight of edges

Linear Programming for Shortest Path

$$\min \sum_{e} f_e w(e)$$

$$\sum_{(u,v)} f_{(u,v)} = \sum_{(v,w)} f_{(v,w)} \forall v \neq s,t$$

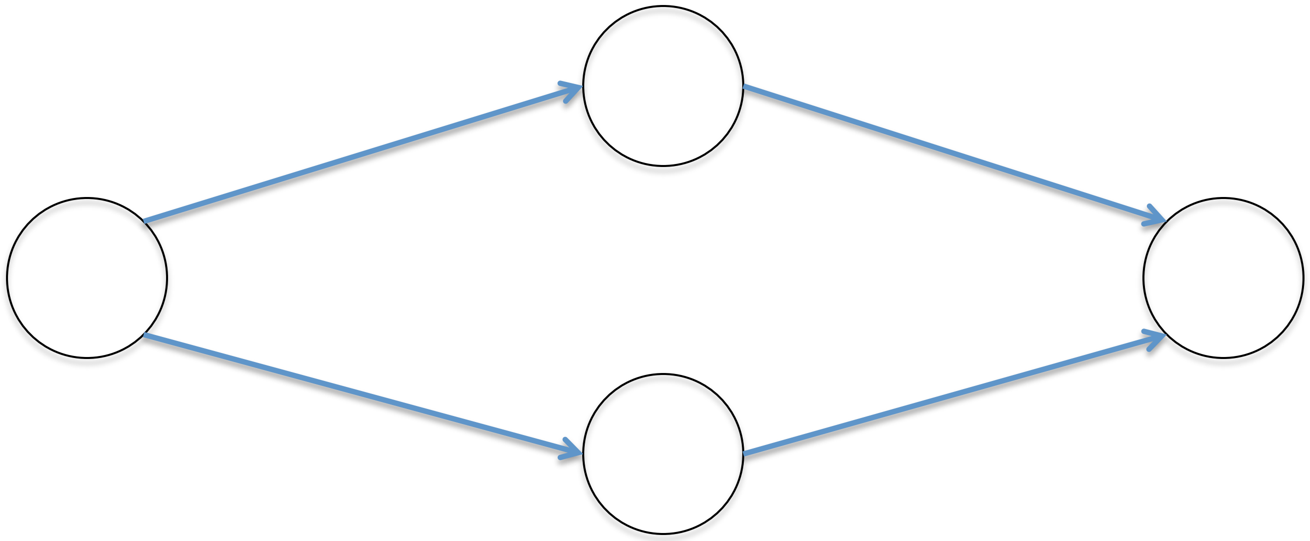
$$\sum_{(s,v)} f_{(s,v)} = 1$$

$$f_e \geq 0$$

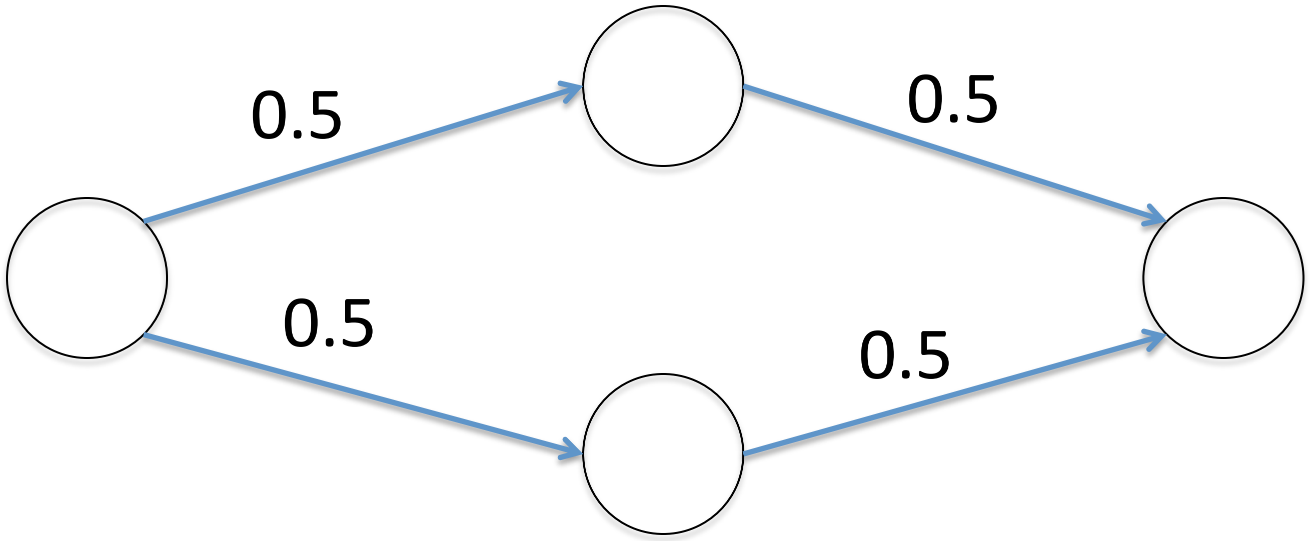
Linear Programming for Shortest Path

- Any path from s to t represents an integer solution to this problem
- Objective function evaluated on such a path is just equal to the weight of the path
- Will the linear program give integer solution?

Linear Programming for Shortest Path



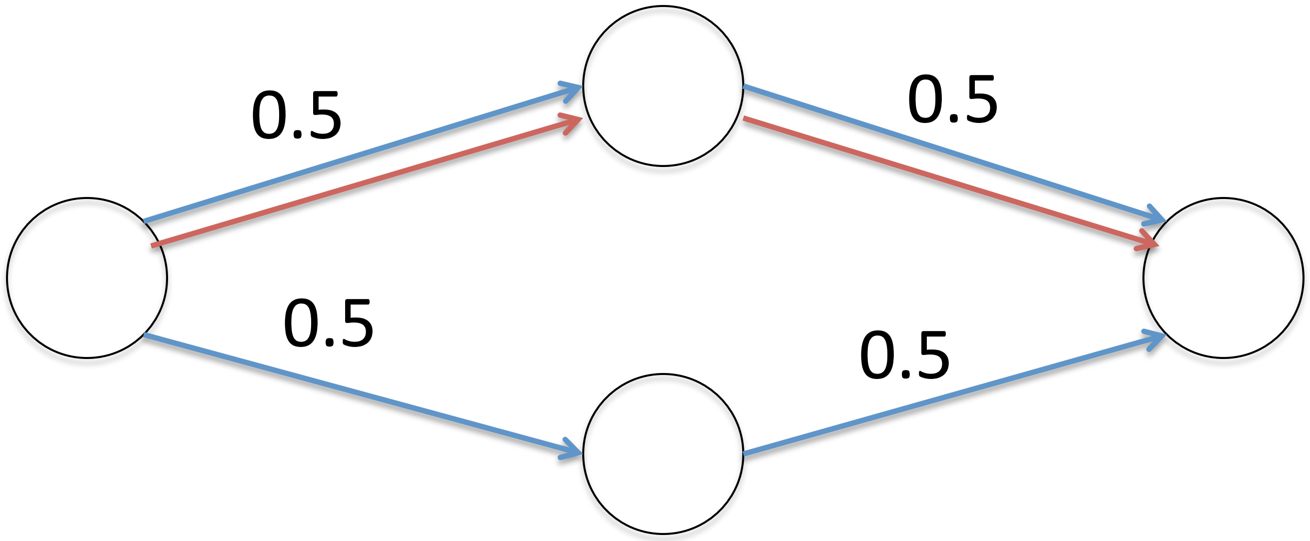
Linear Programming for Shortest Path



Linear Programming for Shortest Path

- In general, solution to linear program does not give us a path
- Solution: take any path from s to t using only edges with non-zero flow

Linear Programming for Shortest Path



Linear Programming for Shortest Path

- Proof of correctness:
  - Suppose we have optimal flow F. Let C_F be the cost
  - Let P be some path from s to t using only edges with non-zero flow.
  - Let C_P be the cost of the flow obtained by sending 1 unit of flow along the edges of P
  - Claim: $C_P \leq C_F$

Linear Programming for Shortest Path

- Claim: $C_P \leq C_F$
  - Proof: Let r be the minimum amount of flow in F along any of the edges in P
  - Subtract r from the flow along each edge in P
  - $C_{F'} = C_F - r\, C_P$
  - The size of the flow F' is 1-r. Multiply the flow in each edge by 1/(1-r)
  - $C_{F''} = (C_F - r\, C_P)/(1-r)$

Linear Programming for Shortest Path

- Claim: $C_P \leq C_F$
  - F'' is a flow of size 1 with cost $C_{F''} = (C_F - r\, C_P)/(1-r)$
  - Therefore, $C_{F''} \geq C_F$

Linear Programming for Shortest Path

$$C_F \le \frac{C_F - rC_P}{1 - r}$$

$$\frac{r}{1-r} C_F \ge \frac{r}{1-r} C_P$$

$$C_F \ge C_P$$

Linear Programming for Shortest Path

- Proof of correctness:
  - $C_P \leq C_F$, so $C_P = C_F$
  - Therefore, the path P is also an optimal solution to the linear program
  - Therefore, it must be the shortest path

Reductions

- Reductions allow us to solve one problem using algorithm for another problem
- Reductions can also be used to show the impossibility of a good algorithm

Reductions

- Suppose we have some really hard problem A, and we don't think we can solve A efficiently
- Suppose further that we have a reduction from solving A to solving some other problem B
- What if we had an efficient algorithm to solve B?

Applications

- Complexity Theory:
  - To prove that we can't solve some particular problem A efficiently, we often come up with some contrived problem B
  - B is defined in a way that allows us to prove that B cannot be solved efficiently
  - We then show a reductions from solving B to solving A, thus showing that A cannot be solved efficiently

Applications

- Cryptography:
  - Prior to the 1970s, cryptographers typically created schemes that resisted known attacks
  - What about attacks that we haven’t though of yet?
  - Goal of modern cryptography: prove no such attacks exist

Applications

- Cryptography:
  - Unfortunately, we have not been able to prove that any useful scheme is secure against all attacks
  - Instead, we start with some hard problem (e.g. factoring integers)
  - We show that if an adversary can break our scheme, they can solve the hard problem
  - Thus, if we assume the problem cannot be solved efficiently, no adversary can break our scheme efficiently