CS 161: Design and Analysis of Algorithms

Greedy Algorithms 2: Minimum Spanning Trees

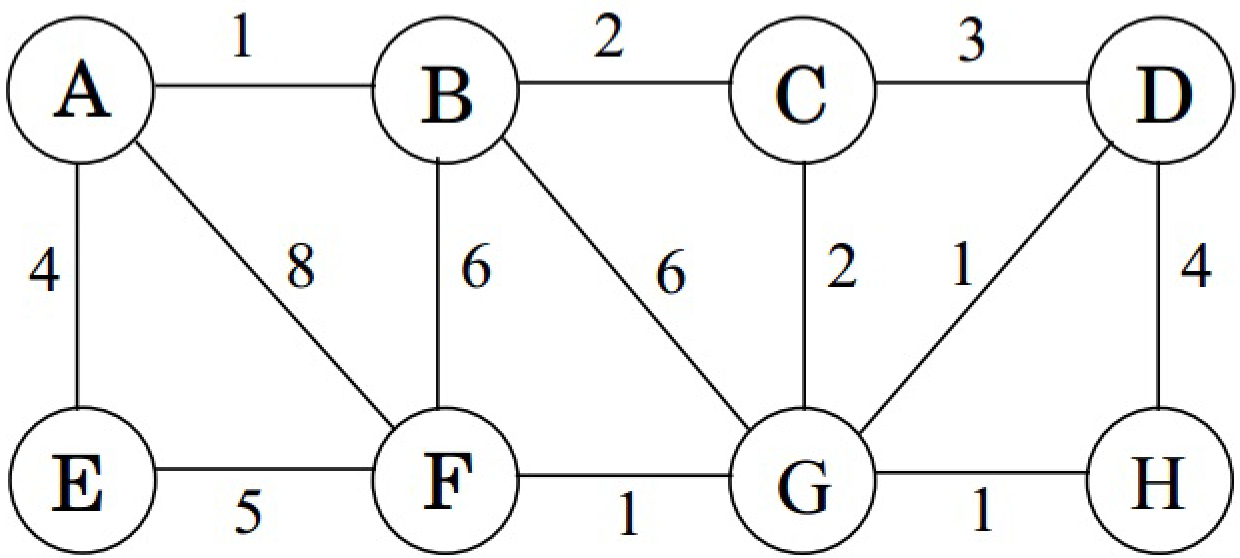
- Definition
- The cut property
- Prim’s Algorithm
- Kruskal’s Algorithm
- Disjoint Sets

Tree

- A tree is a connected graph with no cycles

Minimum Spanning Tree

- Given a weighted undirected graph G=(V,E) with non-negative weights, find a set of edges that connects the graph with the least total weight.



Minimum Spanning Tree

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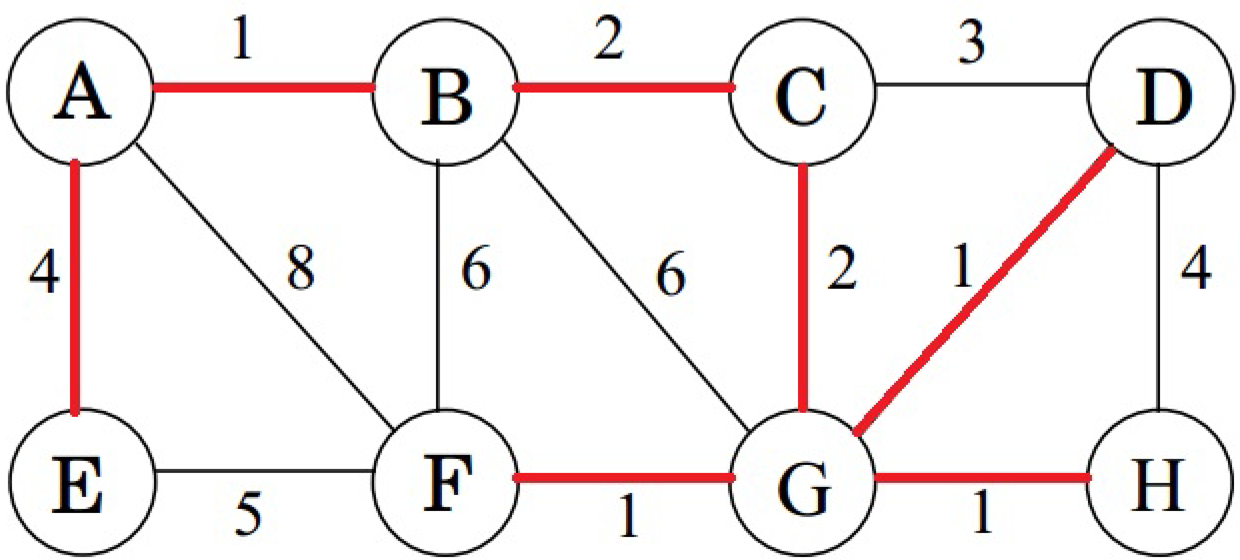


Minimum Spanning Tree

- Uses
  - Networks: If we want to connect a collection of computers by directly connecting pairs, and each connection has a cost, what is the least-cost way of achieving this goal?
  - Approximation: Used in approximation algorithm for Traveling Salesman Problem

Properties

- Why a tree?
- Property 1: Removing an edge in a cycle will not disconnect a graph
  - If two nodes were connected through the removed edge, still connected by going other direction around cycle.

Properties

- Property 2: A tree on |V| nodes has |E|=|V|-1
  - Build tree one edge at a time. Start with one node, no edges, and repeatedly add one edge, one node until tree is constructed.

Properties

- Property 3: Any connected undirected graph with $|E| = |V|-1$ is a tree
  - If not a tree, there is a cycle. Remove any edge on the cycle. By property 1, still connected.
  - Continue until no cycles. The result is a tree, so $|E'| = |V|-1 = |E|$. Thus, no edges were actually removed

Properties

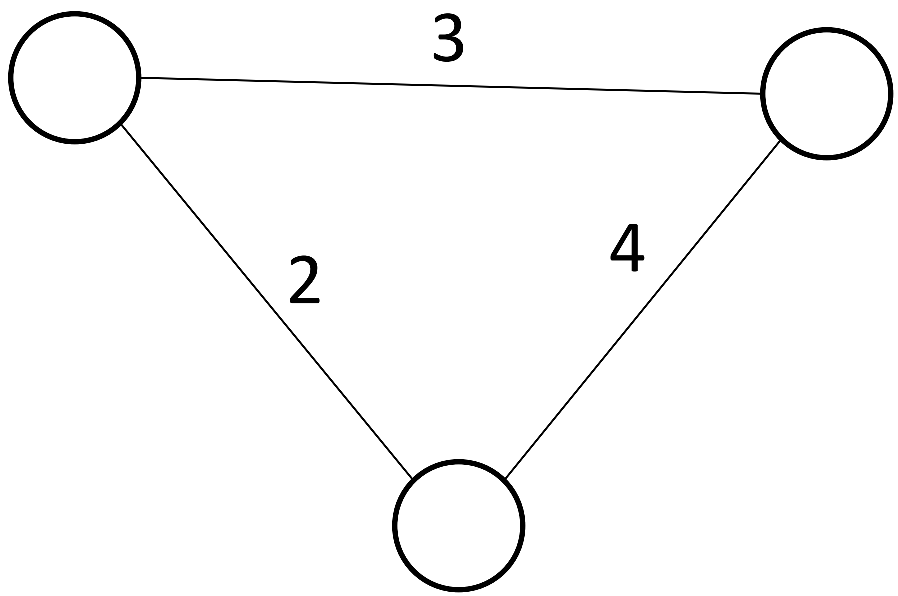
- Property 4: An undirected graph is a tree if and only if there is a unique path between any pair of nodes
  - Trees are connected, so at least 1 path. If two paths, union would contain cycle
  - If not a tree, either disconnected or contains cycle. If disconnected, there are two nodes with no paths. If cycle, any two nodes on cycle have two paths

First Attempt: Unweighted Graphs

- If all weights = 1 (equivalent to unweighted graph), any tree is an MST
- Do DFS, marking tree edges
- $O(|V|+|E|)$

Second Attempt: Weighted Graphs

- Replace each edge with weight w with w edges and w-1 nodes



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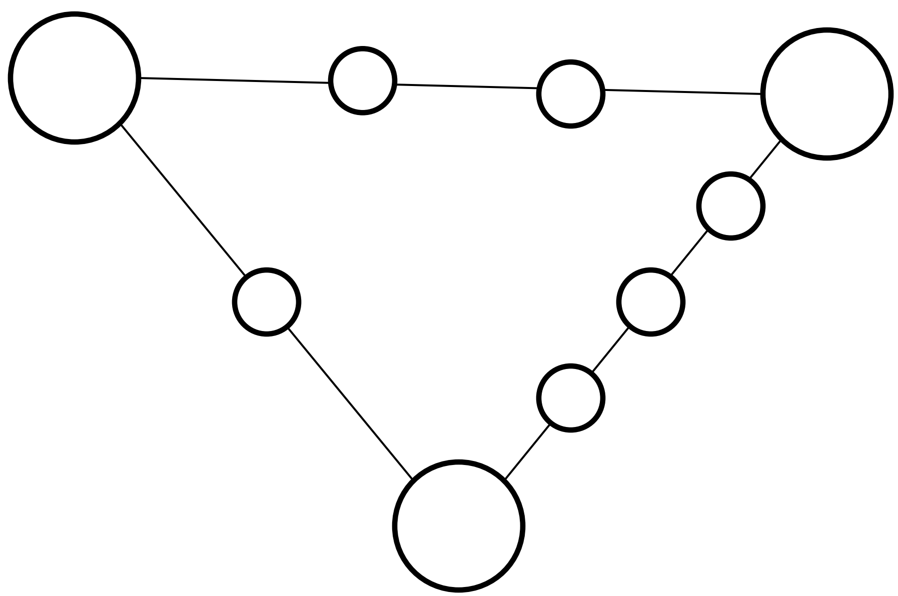


Second Attempt: Weighted Graphs

- Let W be total edge weight
- $|E'| = |E|+W$
- $|V'| = |V|+W-|E|$
- $O(|V'|+|E'|)=O(|V|+W)$
- Inefficient if W large

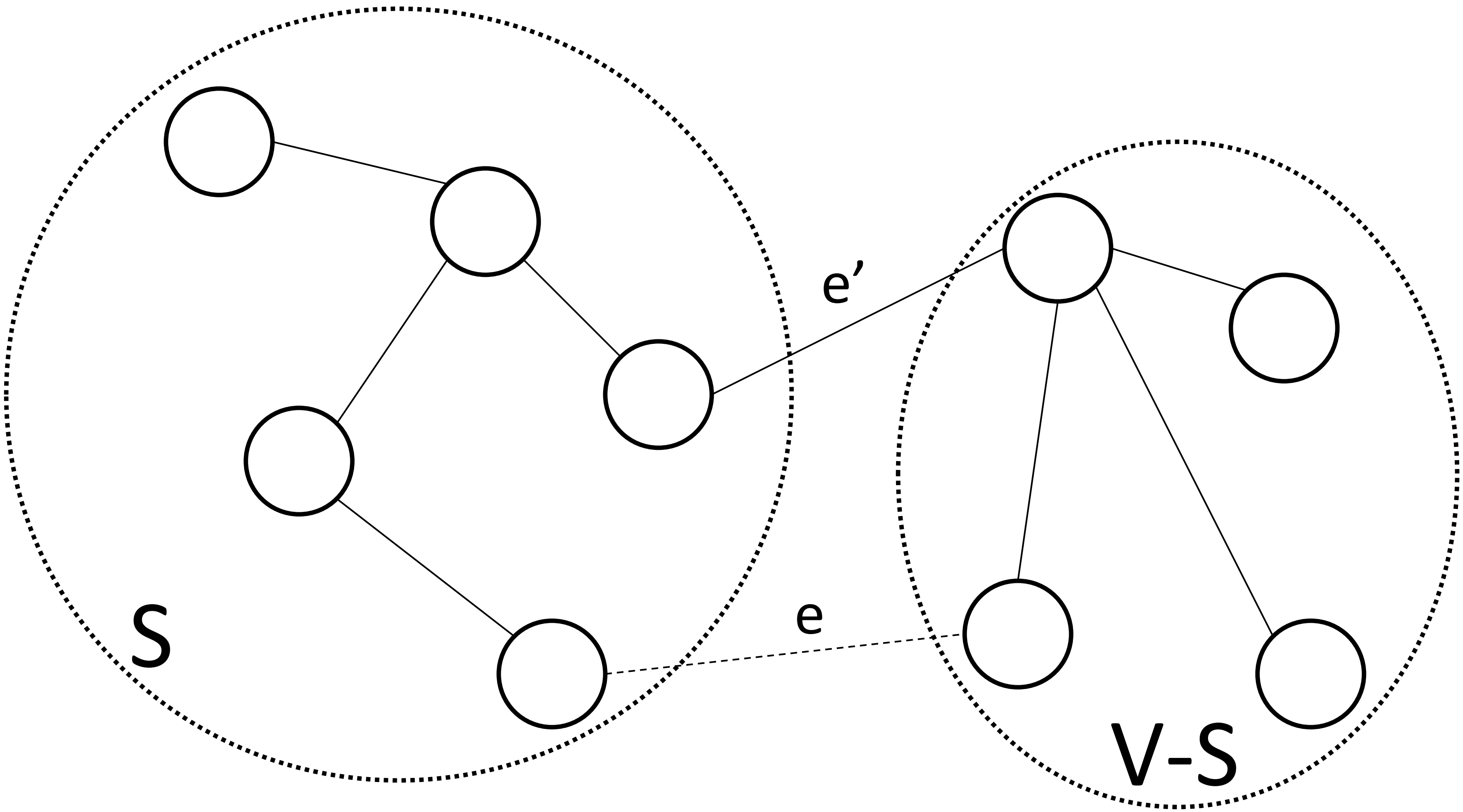
Properties

- A cut of a graph G=(V,E) is a pair (S,V-S)
- Property 5: The Cut Property
  - Let X be a set of edges that are part of some MST of G=(V,E), and (S,V-S) be a cut that X does not cross. Let e be the lightest edge across this cut. Then X U {e} is part of some MST

Proof of Cut Property

- X is part of some MST T
- If e is part of T, then done
- Otherwise, add e to T
- Since T was a tree, adding e creates a cycle.
- Since e crosses the cut, there must be another edge e’ on the cycle that crosses the cut

Proof of Cut Property



Proof of Cut Property

- Let T’ be the graph obtained by adding e to T and removing e’
- T’ is a tree:
  - Since T was connected, and e’ was a cycle edge, T’ still connected
  - T’ has |E’|=|V|-1, so by Property 3, must be a tree

Proof of Cut Property

- weight(T’) = weight(T)+weight(e)-weight(e’)
- Since T was an MST, must have weight(T’)≥weight(T)
- This means weight(e)≥weight(e’)
- But e was chosen to be lightest edge across cut
  - weight(e)≤weight(e’)
- Thus weight(e)=weight(e’), and weight(T’)=weight(T)
- T’ is an MST

Idea

- From any partial solution X to MST problem with k edges, can get solution X’ with k+1 edges as follows:
  - Pick cut that X does not cross
  - Let e be lightest edge across cut
  - Add e to X

Third Attempt: Prim’s Algorithm

- X is always a tree (never disconnected)
- Cut (S,V-S): S are nodes connected by X
- For each u not in S, must keep track of
  - cost(u): weight of lightest edge from u into S
  - prev(u): the corresponding node in S
- Repeatedly find u with lowest cost(u), and add edge (prev(u),u)

Third Attempt: Prim's Algorithm

- How to keep track of cost and prev?
- If set correctly before adding edge (prev(u),u):
  - For every neighbor v not in S, cost(v) only changes if weight(u,v) < cost(v), so update accordingly
  - If v is not a neighbor, cost(v) does not change

Third Attempt: Prim's Algorithm

- How to find lightest node across edge?
- Heap with cost values as keys

Third Attempt: Prim's Algorithm

- Pick some initial node v
- Set cost(v) = 0, cost(u) = ∞ for u $\neq$ v
- Create heap q with all nodes, ordered by cost
- While q is not empty
  - u = q.deletemin()
  - cost(u) = 0
  - For each (u,w) in E, if cost(w) > weight(u,w):
    - prev(w) = u
    - cost(w) = weight(u,w)
- Output edges (prev(u),u) for all u $\neq$ v

Running Time

- Same as Dijkstra's algorithm!
- $O((|E|+|V|)\log |V|)$ for heaps
- $O(|V|\log |V| + |E|)$ for Fibonacci Heaps
- Is it possible to do any better?

Fourth Attempt: Kruskal's Algorithm

- Repeatedly add lightest edge that does not create cycle
- Same as adding lightest edge across any cut that partial solution does not cross:
  - If adding e = (u,v) does not create cycle, then u and v cannot be connected
    - Let S be component containing u, V-S everything else
  - If adding (u,v) does create a cycle, u and v must have already been connected
    - Any cut separating u and v must be crossed by partial solution

Fourth Attempt: Kruskal's Algorithm

- How to find lightest edge that doesn't create cycle:
  - First sort them by weight
  - Have different sets of nodes that represent different components
  - Go through edges, checking if endpoints are in different sets
  - Combine the two sets into one

Disjoint Sets

- Want the following operations:
  - makeset(x): makes a set containing x
  - union(x,y): unions the two sets
  - find(x): returns the set containing x

Idea 1: Linked Lists

- Each set is represented by a linked list. Additionally, we store, for each value x, a pointer set(x) to the list containing that value
- makeset(x) = create new list L containing x, set(x) = L: O(1)
- find(x) = set(x): O(1)
- union(x,y) = link lists together and change set pointers: O(k) where k is the size of the smaller set

Idea 1: Linked Lists

- Claim: Any sequence of k union operations takes O(k log k) time
  - Proof: Running time constrained by number of updates to set pointers
  - At most 2k nodes affected, so largest set is at most 2k
  - Every time set(x) is updated, the size of the set containing x at least doubles
  - Therefore, number of updates to set(x) is at most

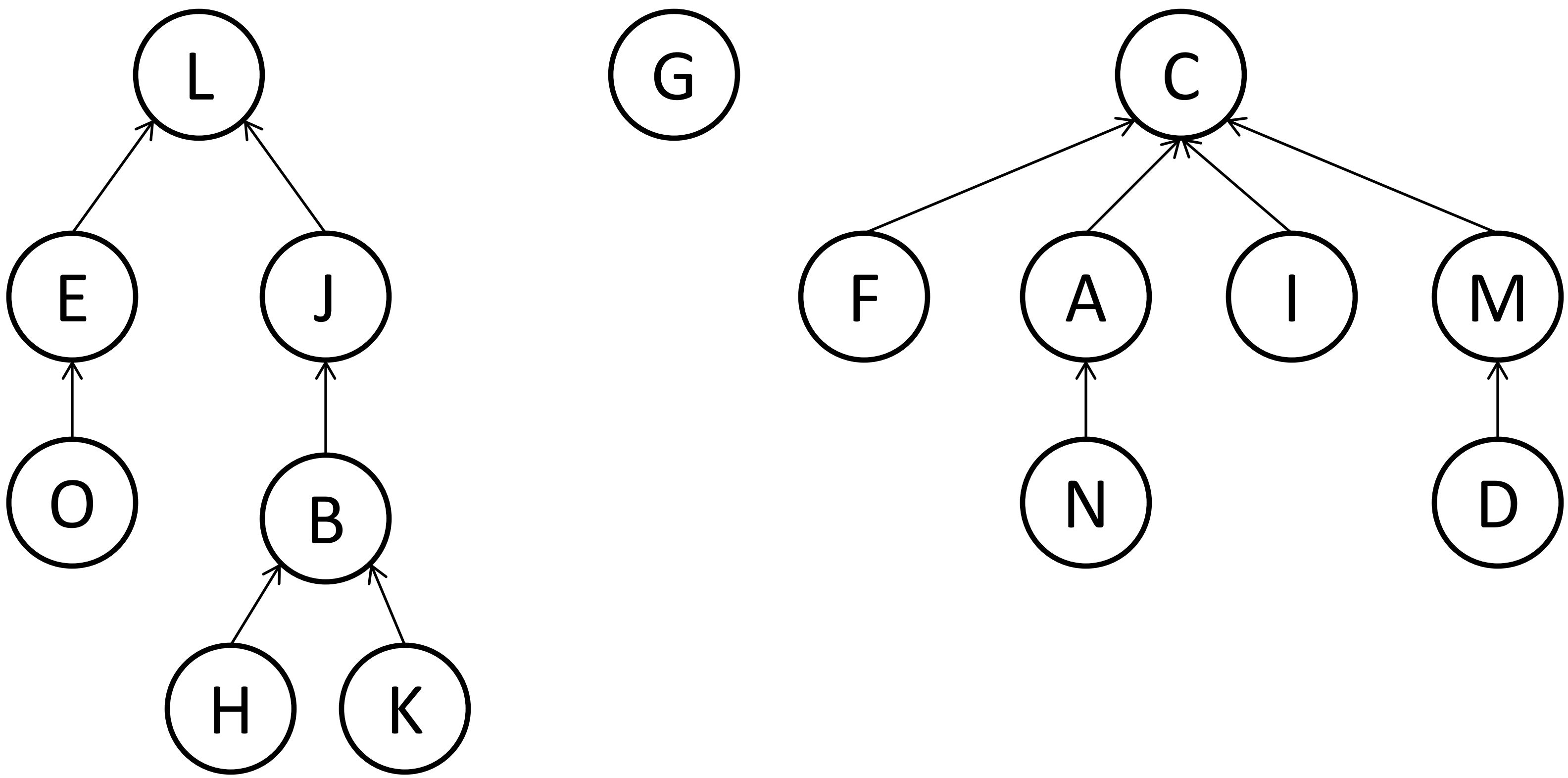
    log 2k
  - Total number of updates at most 2k log 2k = O(k log k)

Idea 1: Linked Lists

- makeset: O(1)
- find: O(1)
- union: O(log n) amortized

Idea 2: Trees

- Each set represented as directed tree
- Set identified by root of tree



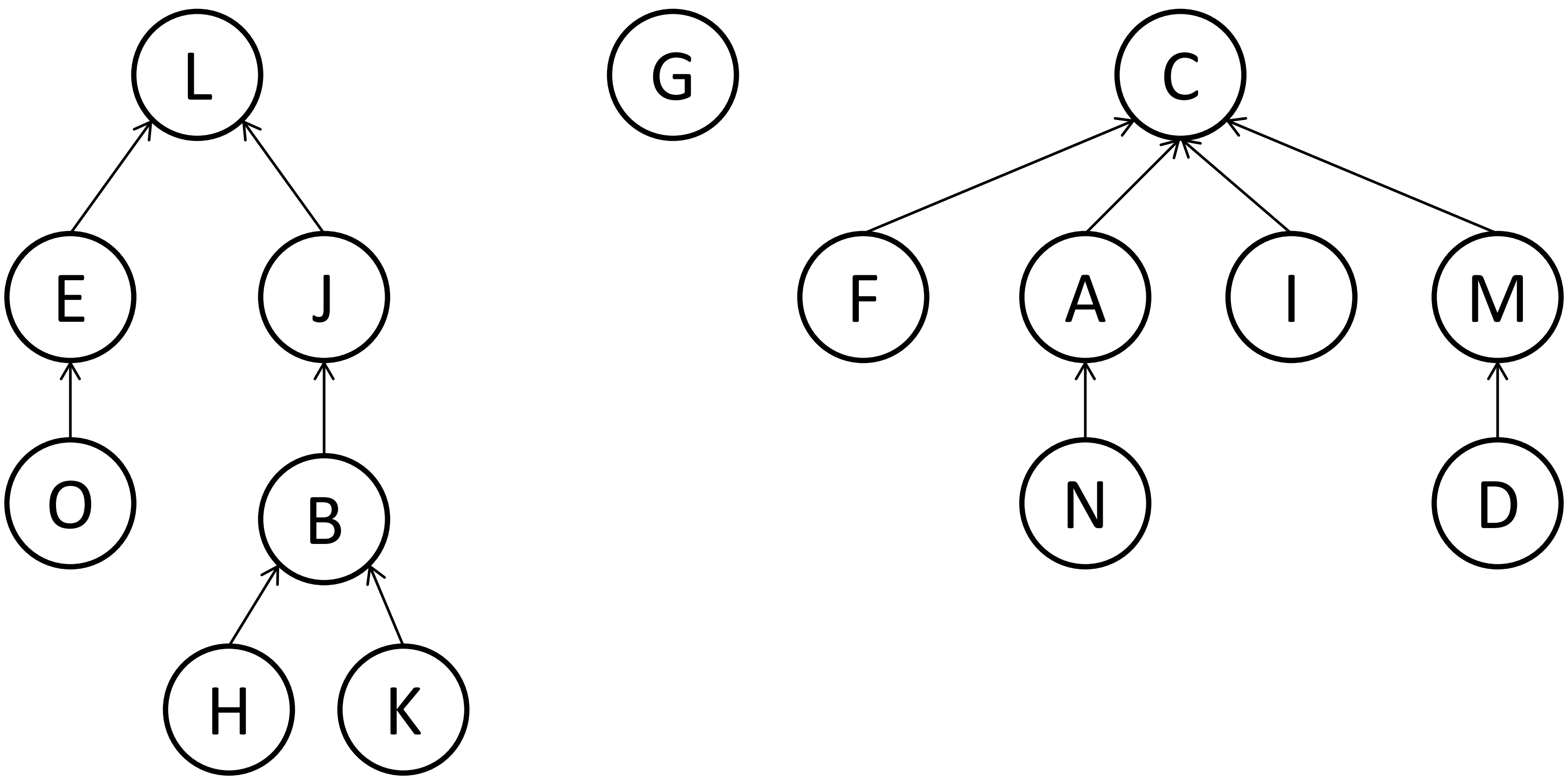
Idea 2: Trees

- Each value x has a parent pointer p(x)
- Root of a tree has p(x) = null
- makeset(x) = make new tree with x as root (i.e. set p(x) = null): O(1)
- find(x) = find root of tree: O(h)
  - Let r = x
  - While p(r) ≠ null, let r = p(r)
  - Output r

Idea 2: Trees

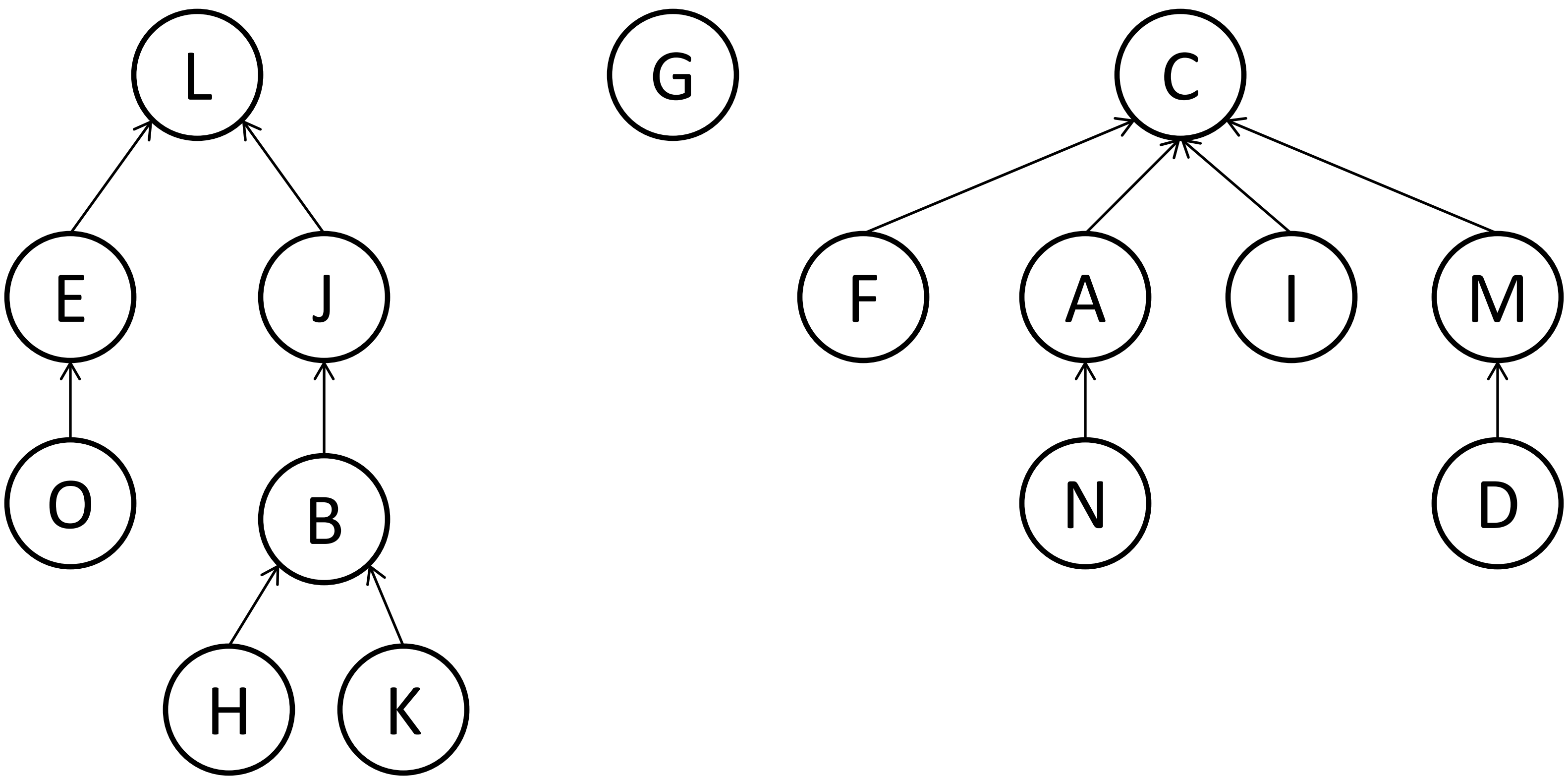
- union(x,y) = set root of one tree to be child of the other: $O(h_1 + h_2)$
  - Let x’ = find(x), y’ = find(y)
  - p(x’) = y’ or p(y’) = x’

Idea 2: Trees



Idea 2: Trees

- makeset(P)



Idea 2: Trees

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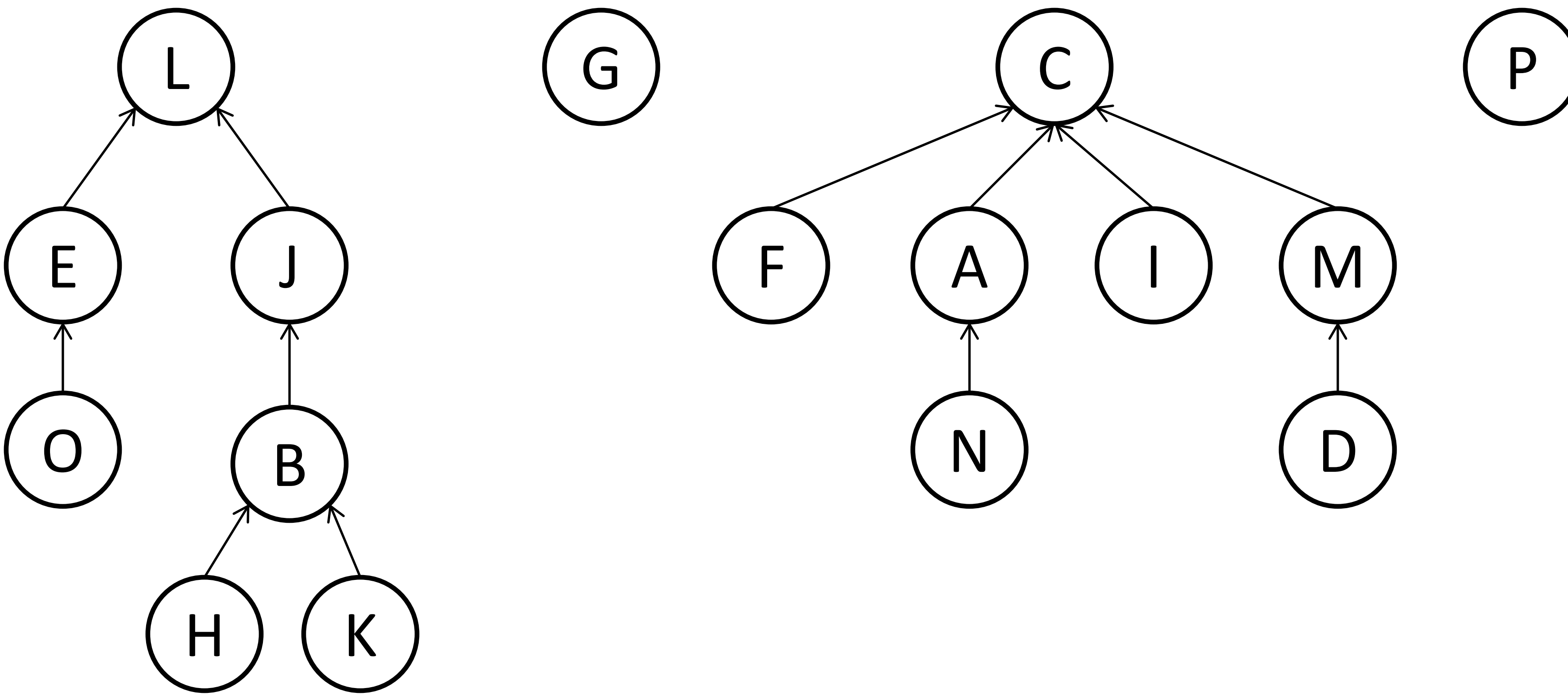
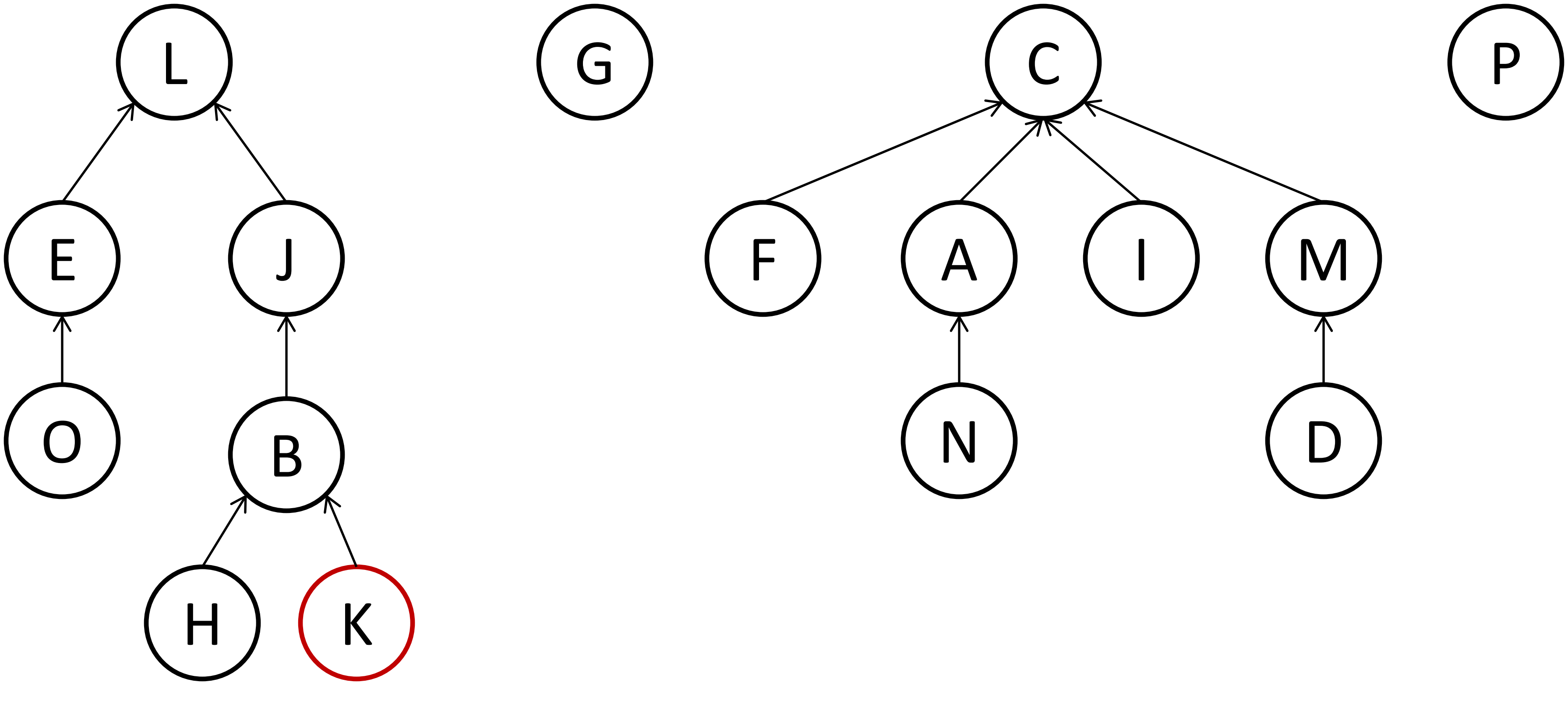


Idea 2: Trees

- find(K)



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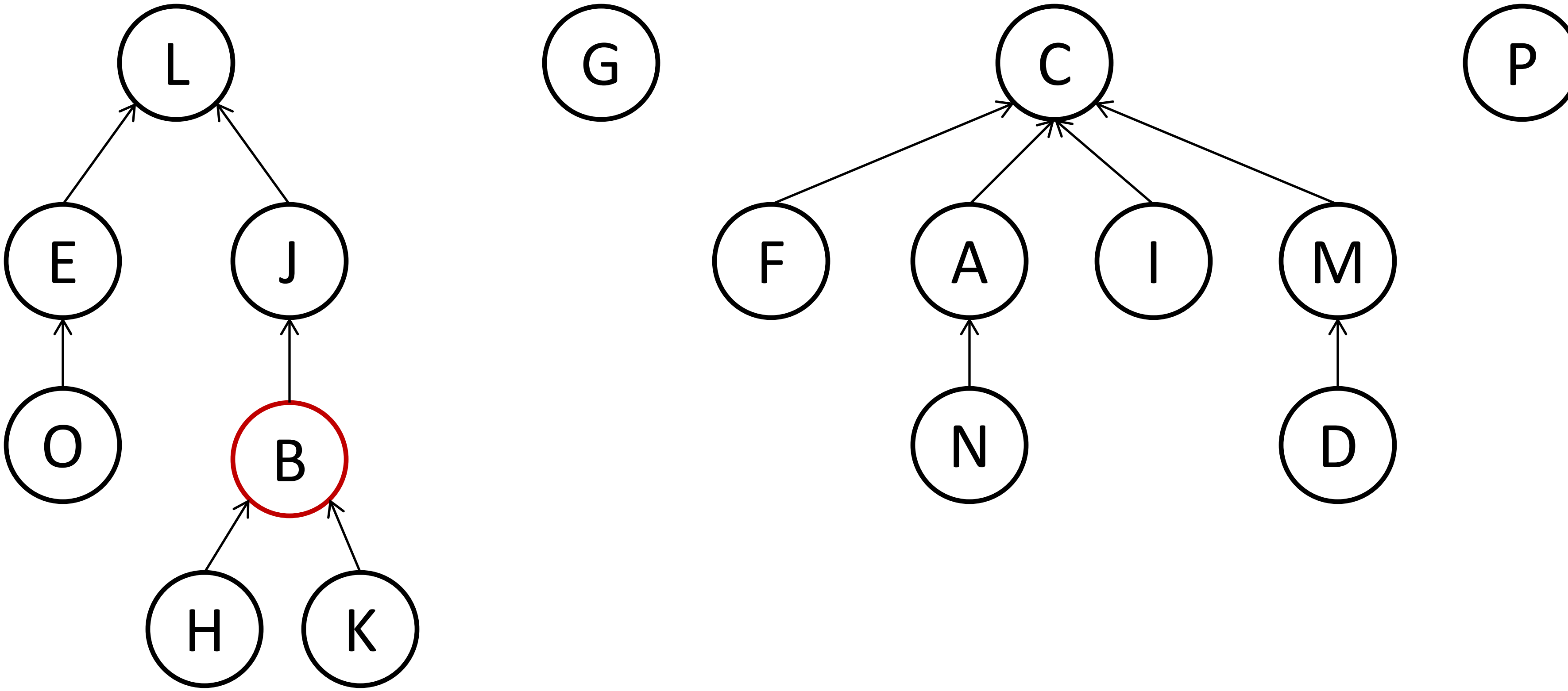

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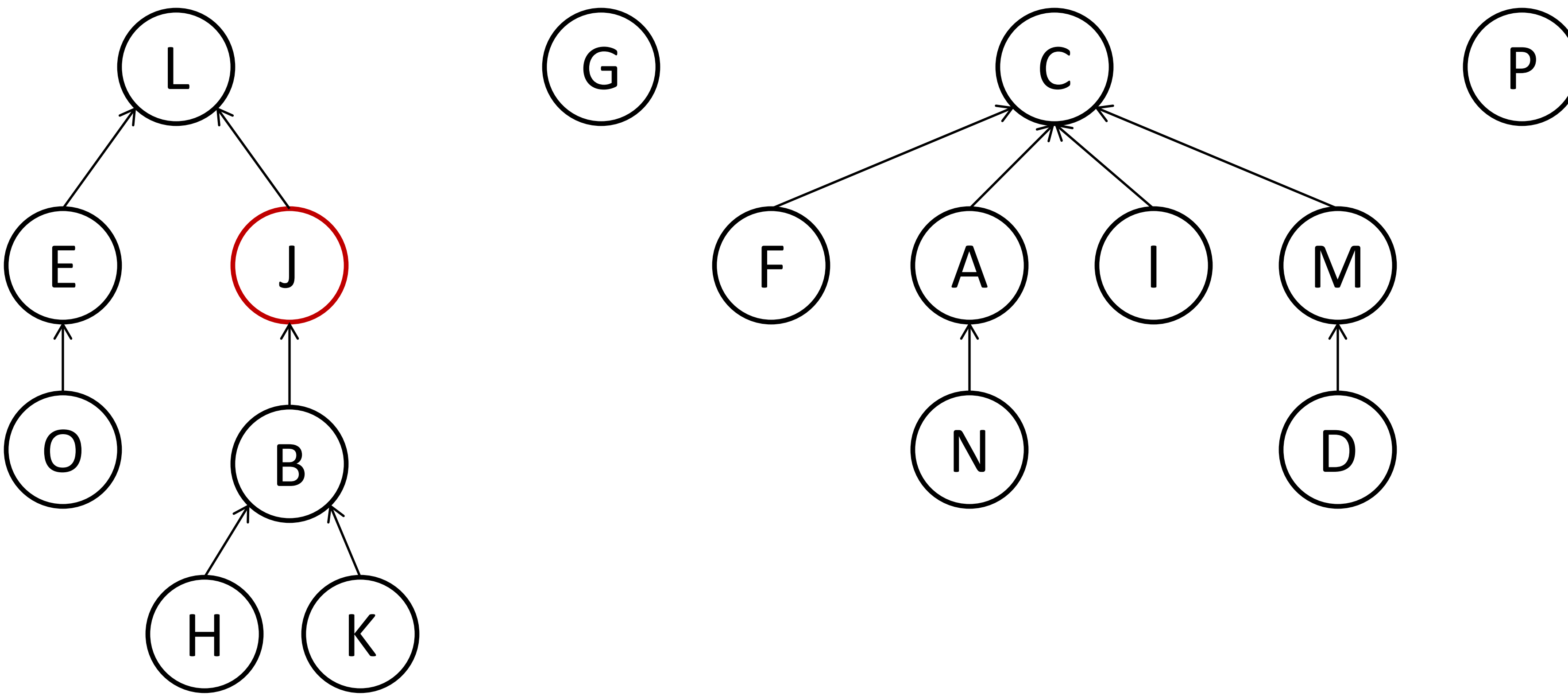

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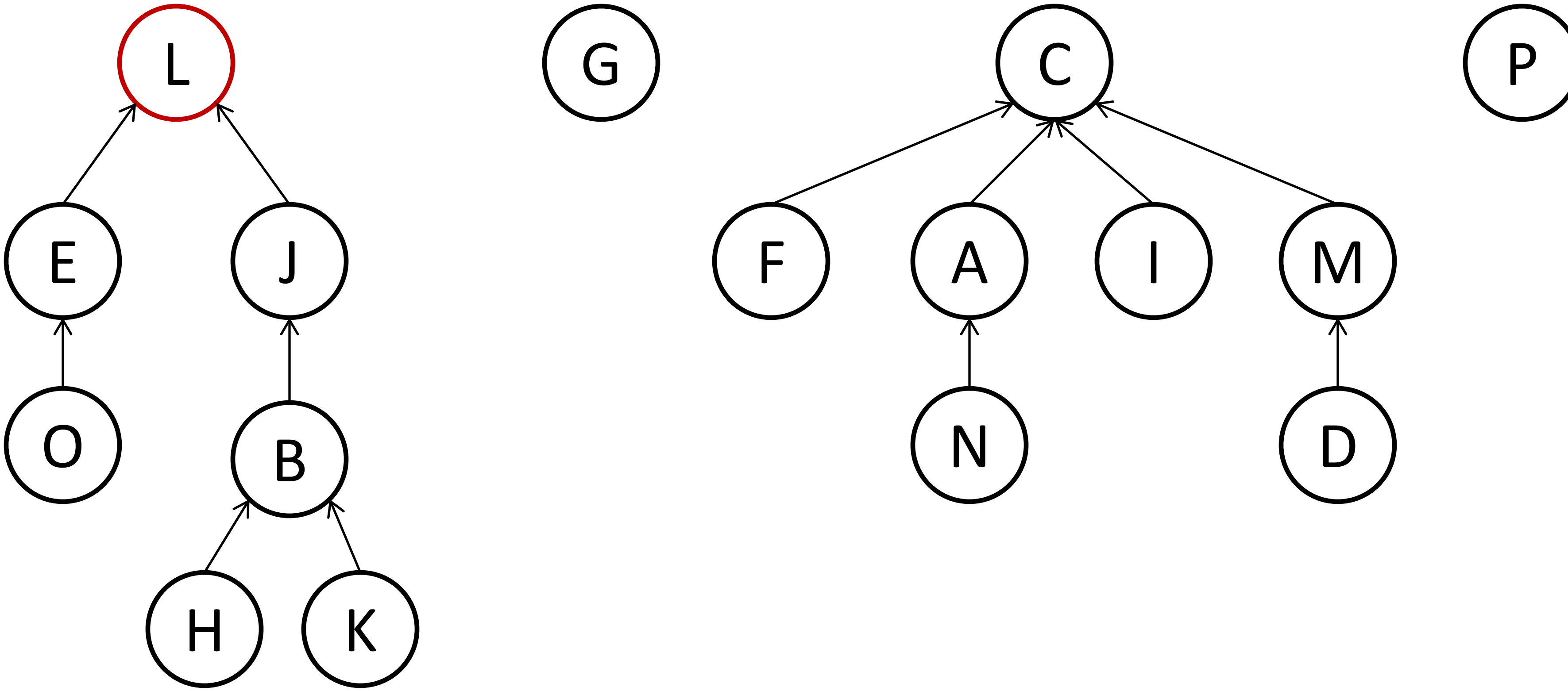
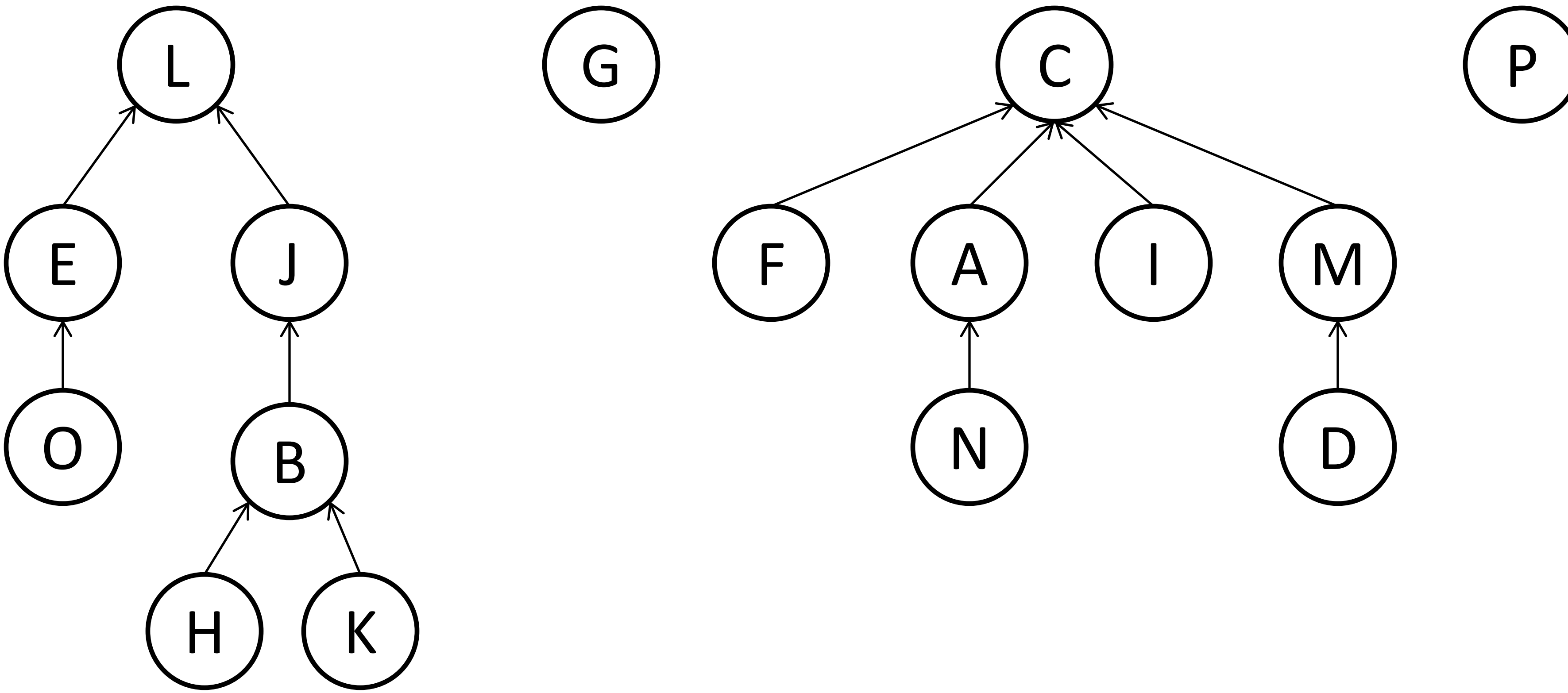


Idea 2: Trees

- Union(K,N)



Idea 2: Trees

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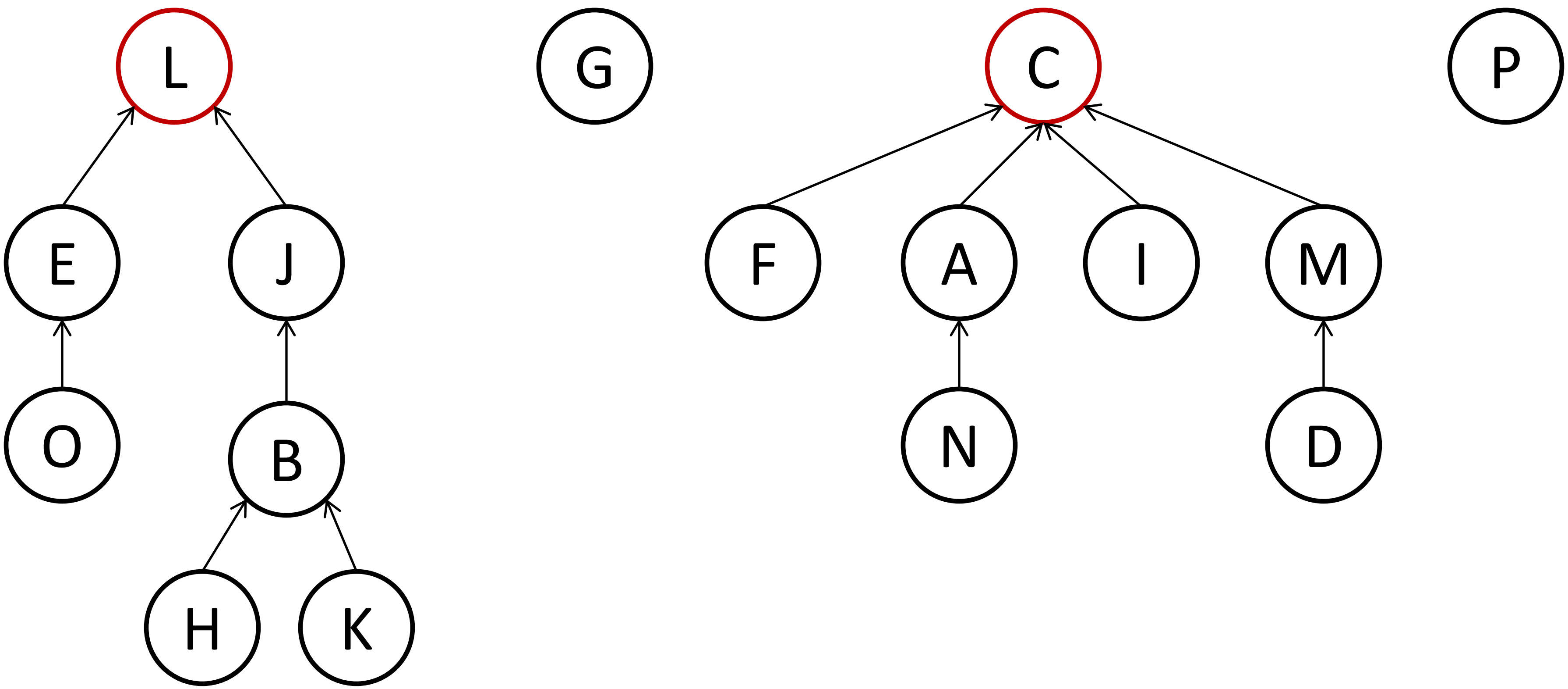
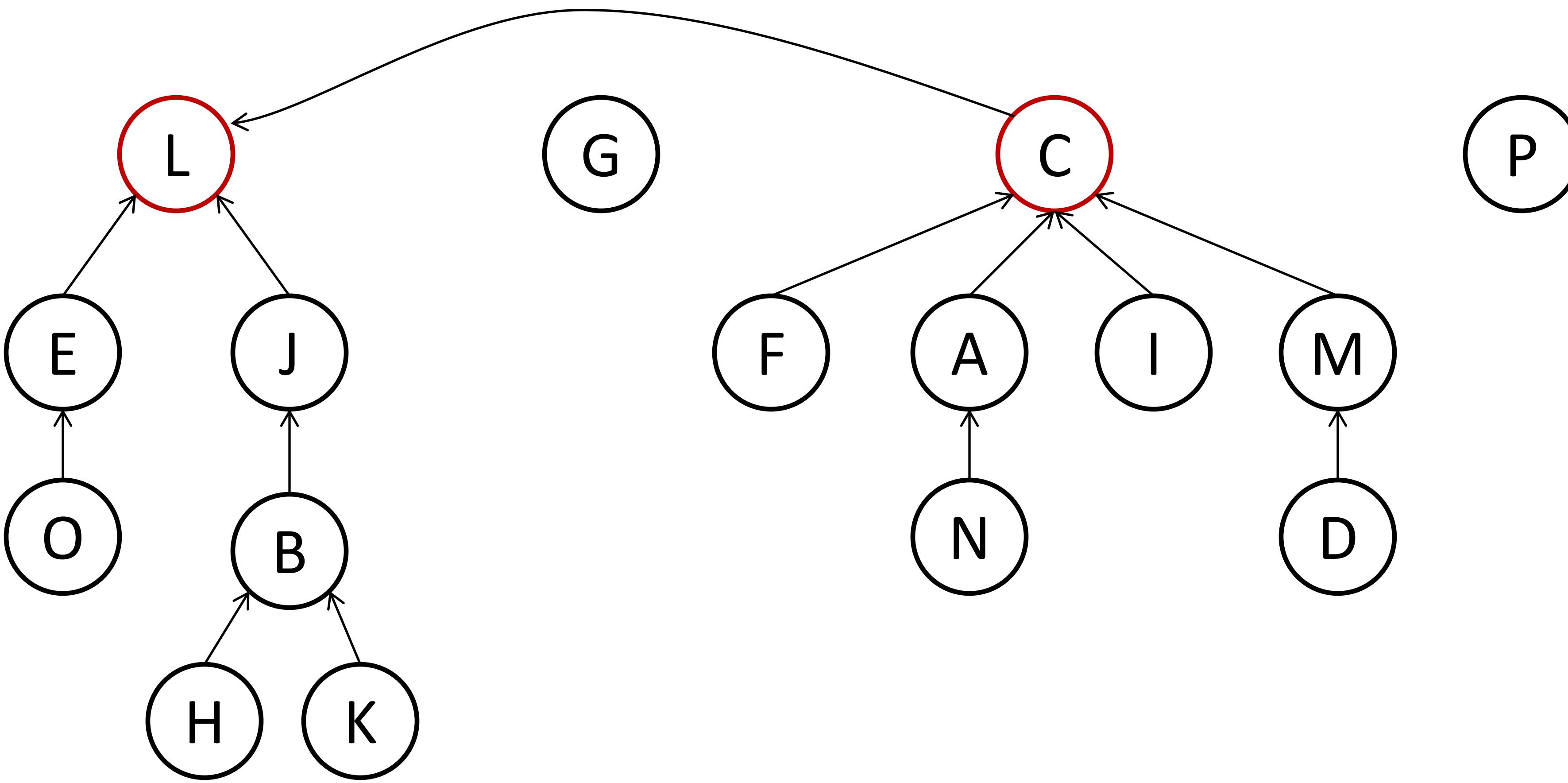


Idea 2: Trees

- Union(K,N)



Idea 2: Trees

- Operation times depend on height of tree, so need to keep trees shallow
- When unioning two sets, make the shorter tree the subtree
- Also keep track of rank(x), the height of the subtree at x - 1

Idea 2: Trees

- makeset(x) = { O(1)
  - Set p(x) = null
  - Set rank(x) = 0

  }
- find(x) = { O(h)
  - Let r = x
  - While p(r) ≠ null, r = p(r)
  - Return r

  }

Idea 2: Trees

- union(x,y) = { $O(h_1 + h_2)$
  - Let x’ = p(x), y’ = p(y)
  - If rank(x’) > rank(y’):
    - p(y’) = x’
  - Else
    - p(x’) = y’
    - If rank(x’) = rank(y’): rank(y’) = rank(y’) + 1

Idea 2: Trees

- Height of tree?
- rank(x) < rank(p(x))
  - Whenever we set p(x) in union, make sure that rank(x) < rank(p(x))
  - We only change the rank of root nodes, so rank(x) never changes.
  - Ranks can only increase.

Idea 2: Trees

- Height of tree?
- rank(x) < rank(p(x))
- Any root of rank k has at least 2^k descendants
  - True before any unions
  - Assume true before union(x,y). Let k_1 and k_2 be the ranks of the root nodes in the two trees
  - Total number of nodes $\geq 2^{k_1} + 2^{k_2}$
  - If new node has rank $k = k_b$ for some b, property holds
  - Otherwise, $k_1 = k_2$, $k = k_1+1$, and property holds

Idea 2: Trees

- Height of tree?
- rank(x) < rank(p(x))
- Any root of rank k has at least 2^k descendants
- There are at most $n/2^k$ nodes of rank k
  - Any such node has at least 2^k descendants
  - No two nodes of rank k can share descendants

Idea 2: Trees

- Height of tree?
- rank(x) < rank(p(x))
- Any root of rank k has at least 2^k descendants
- There are at most $n/2^k$ nodes of rank k
- Maximum rank is at most log n

Idea 2: Trees

- Since maximum height of tree is log n, find and union take O(log n)
- Worse than linked lists implementation
- Next time: We will see how to improve this to almost constant time