CS 161: Design and Analysis of Algorithms

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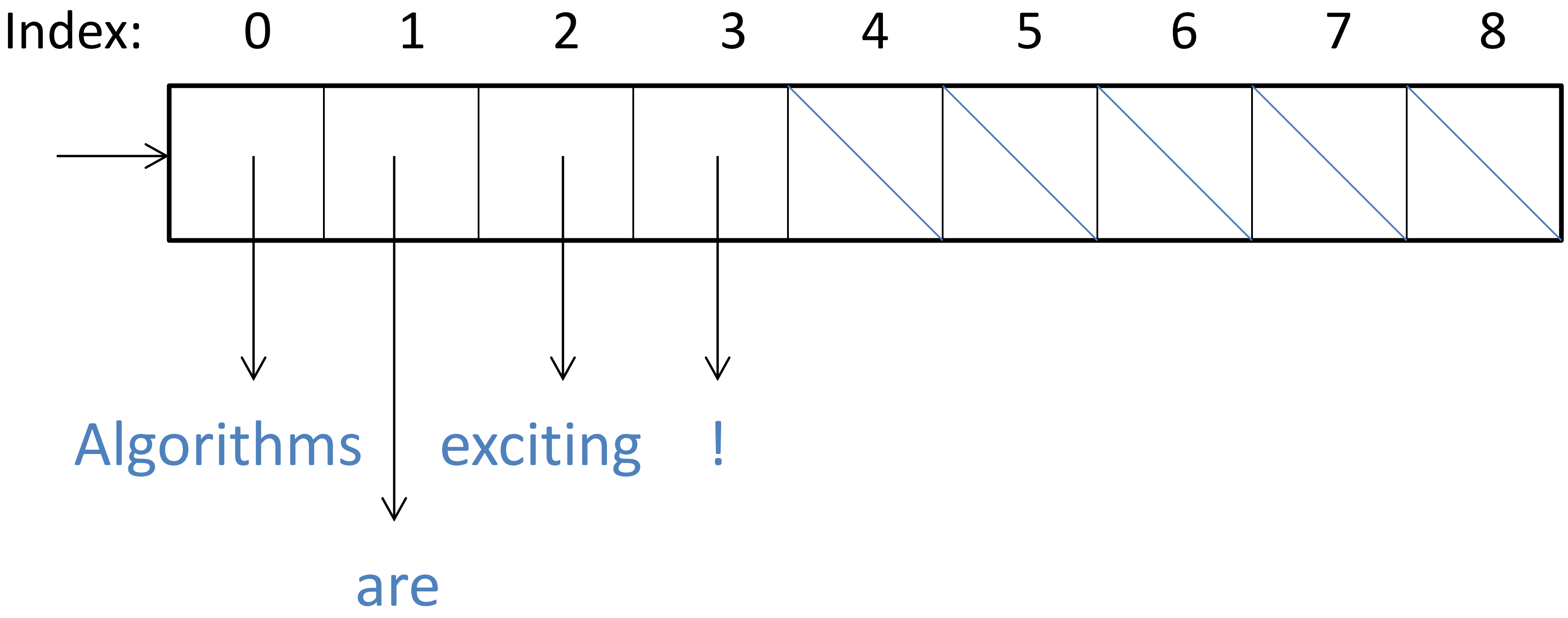
Data Structures I:
Storing Unordered Data

- Arrays
- Linked Lists
- Stacks
- Queues
- Hash Tables

The Problem

- We have a collection of items we would like to store for later retrieval.
- Items are not comparable (i.e. no notion of $x < y$)

Array Lists



Index: 0 1 2 3 4 5 6 7 8

Algorithms

are

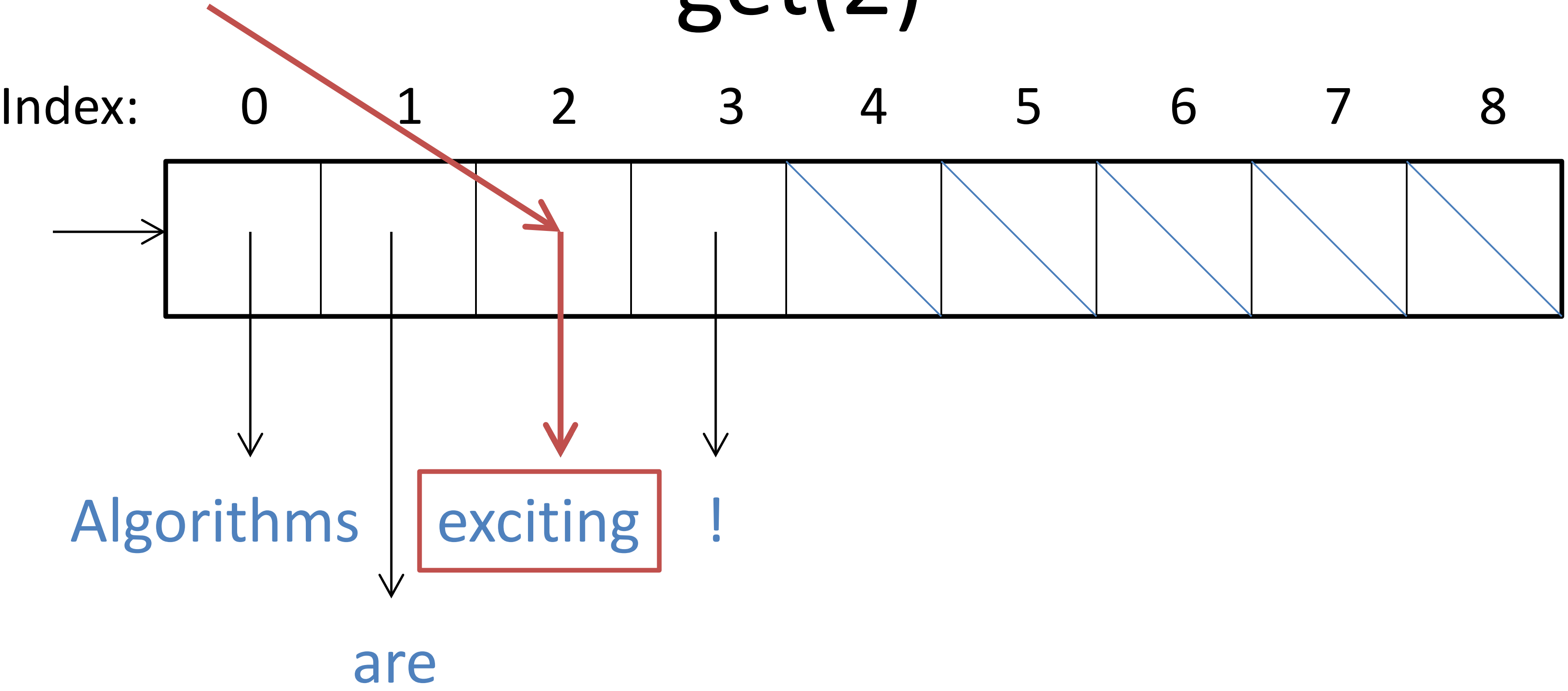
exciting

!

Array Lists

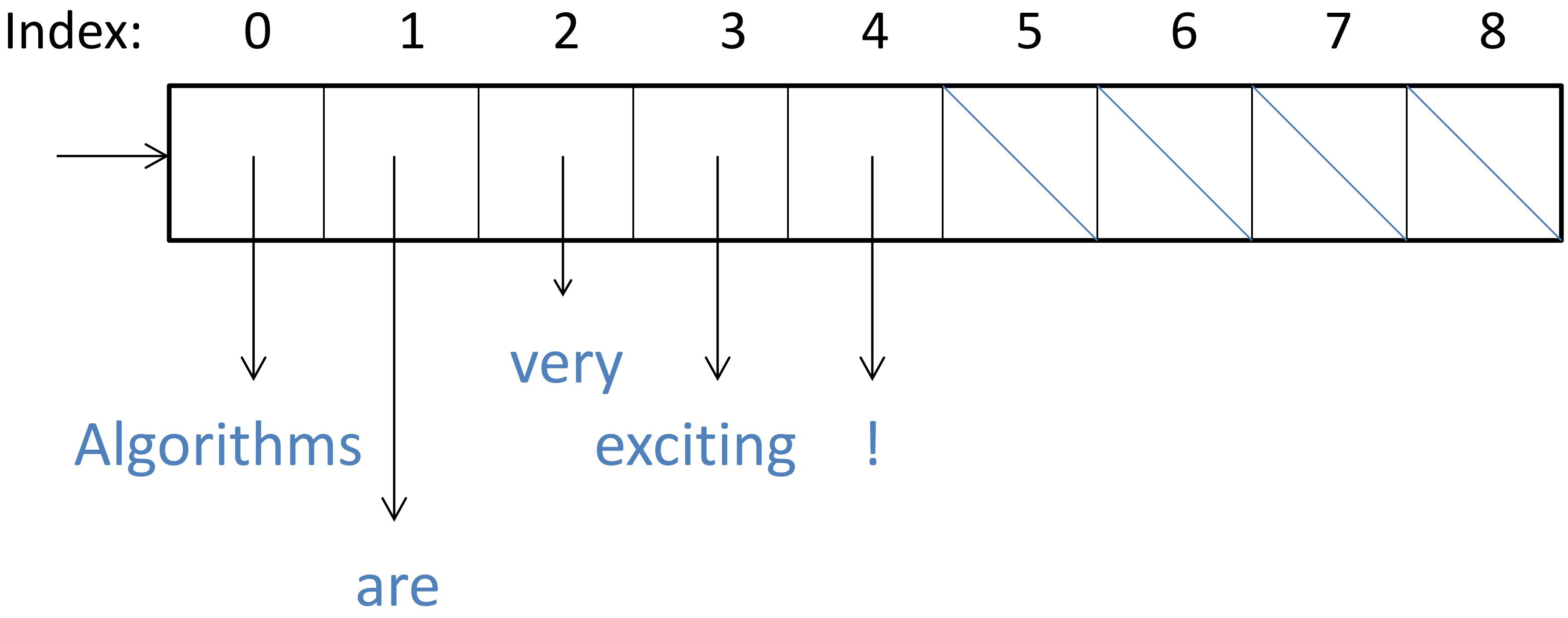
- Operations:
  - get(i) returns the value at index i
  - add(i,x) inserts x at index i
    - All values after index i move down the array
  - remove(i) removes the value at index i
    - All values after index i move up the array

get(2)



- Turns out that the time to retrieve values does not depend on length of array (i.e. O(1) time)

add(2,“very”)



Index: 0 1 2 3 4 5 6 7 8

very

Algorithms

exciting

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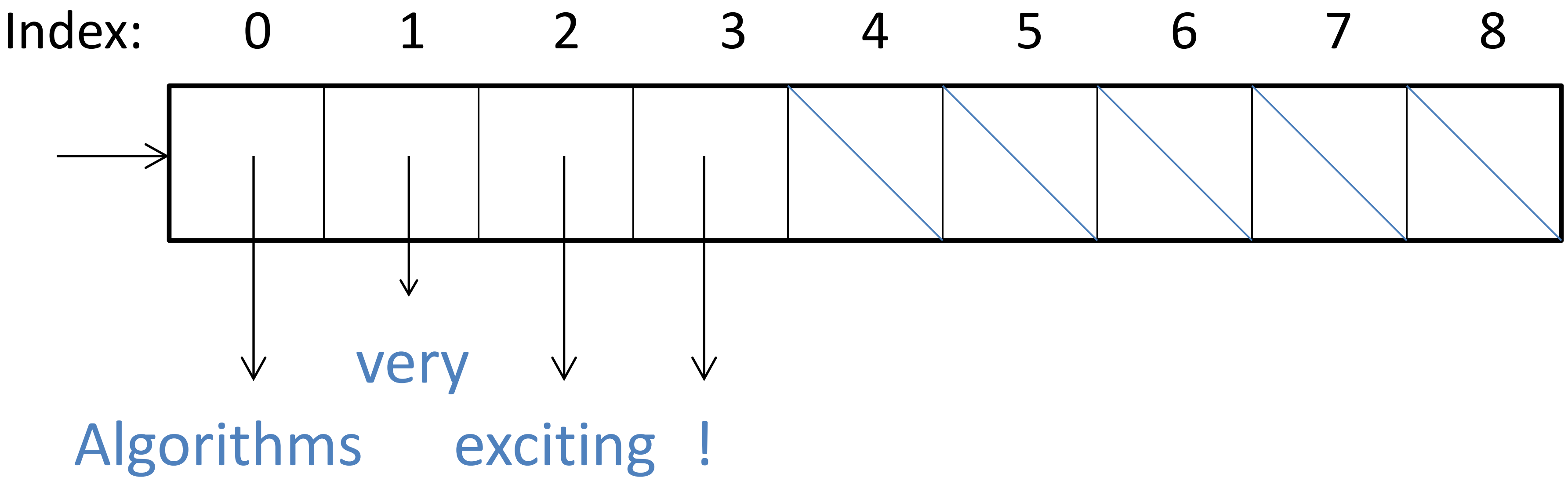
are

- add(i,x) requires time proportional to the number of shifts
  - O(1) time at end of array ***
  - O(n) at beginning

remove(1)



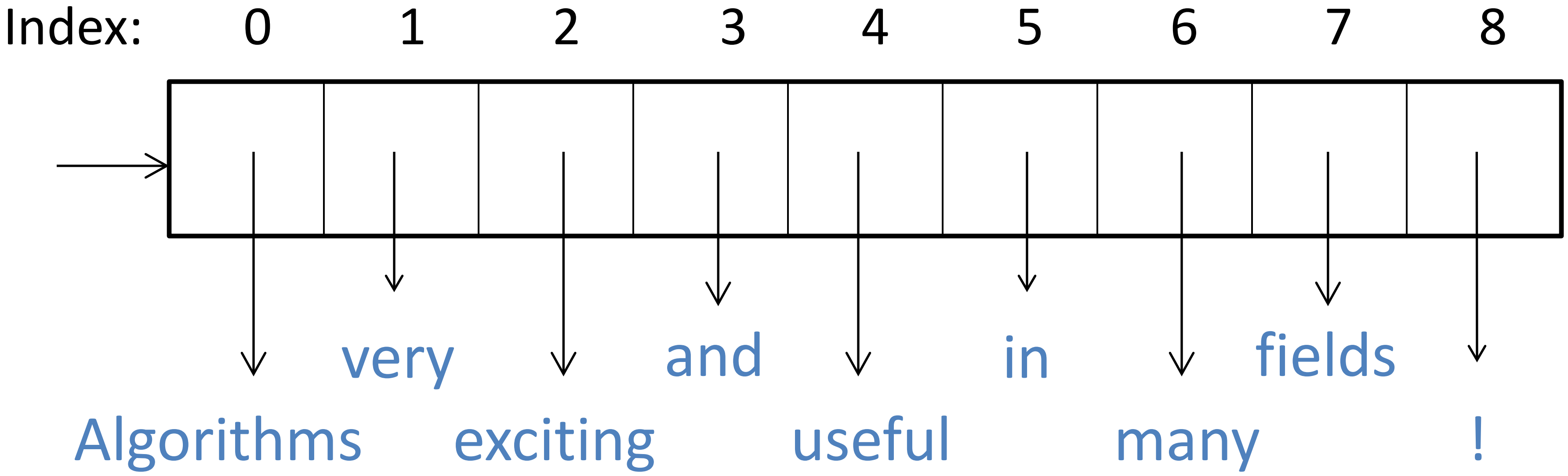
Index: 0 1 2 3 4 5 6 7 8

very

Algorithms exciting !

- remove(i) requires time proportional to the number of shifts
  - O(1) time at end of array
  - O(n) at beginning or middle

Problem!!!



Index: 0 1 2 3 4 5 6 7 8

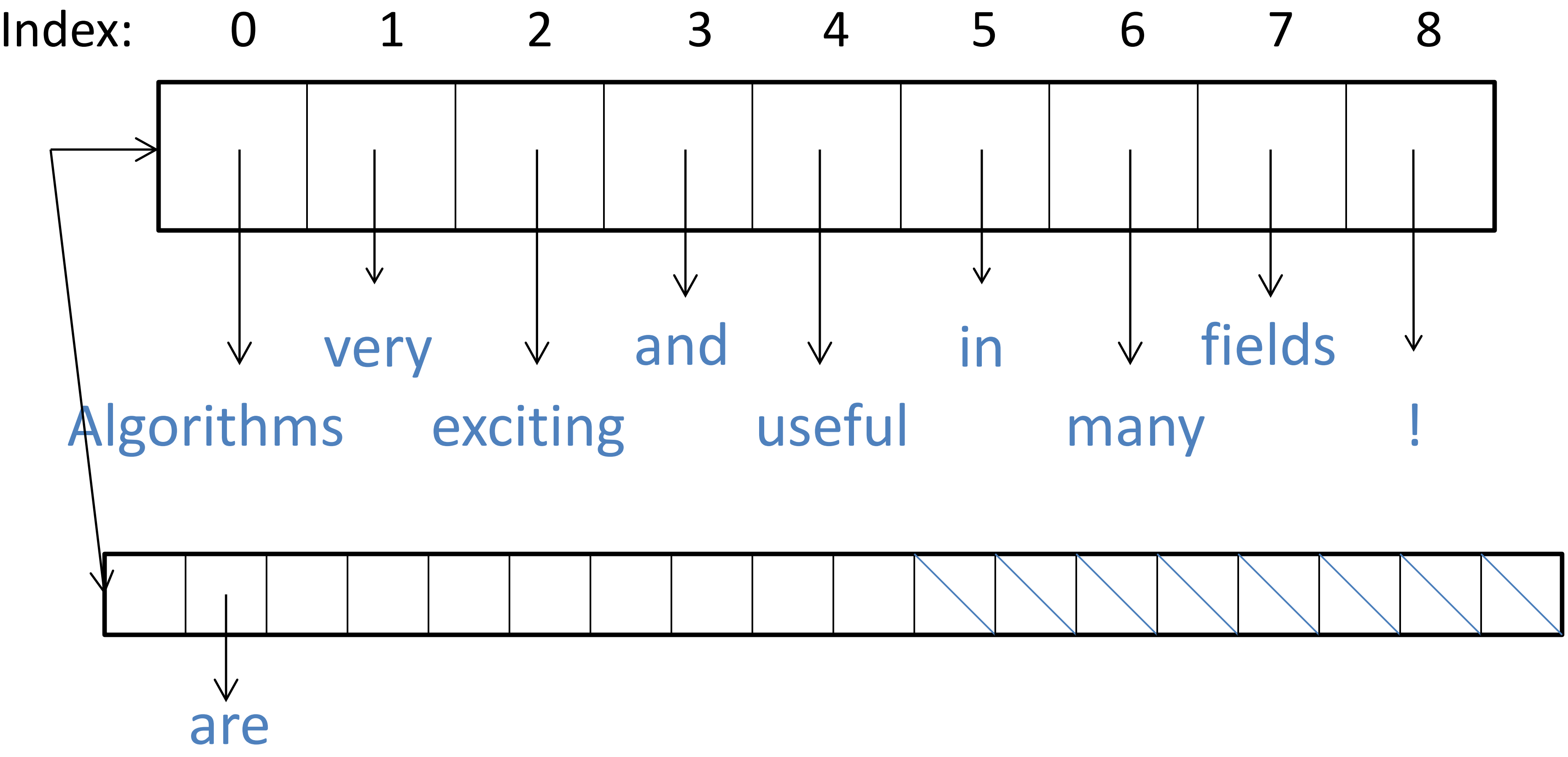
Algorithms very exciting and useful in many fields !

- If we call add(1,“are”), no room to insert!

Solution: Dynamic Arrays

- Automatically grows when space runs out.
- Must create new array and copy
  - If we grow by 1, we will do many copy operations
  - Instead, we grow by doubling the size

Dynamic Arrays



Algorithms very exciting and useful in many fields !

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Amortized Analysis

- Adding to beginning or middle still takes O(n)
- Adding to end now sometimes takes O(n)
  - However, most of the time O(1).
- Can we make any stronger statements than O(n)?
- Amortized Analysis: consider sequence of operations

Amortized Analysis

- What if we add n values to an initially empty array, always adding to the end? Say $n = 2^k$
- Suppose initial size of array is 1, and $n = 2^k$
- After adding all n values, how many copies have we made?
- How much total space has been allocated?

Amortized Analysis

- We double only when the array has size 2^i ($i < k$), and the array is full.
  - Number of copies: 2^i.
  - Amount of space allocated: 2^{i+1}.
- Total copies:
- Total space:

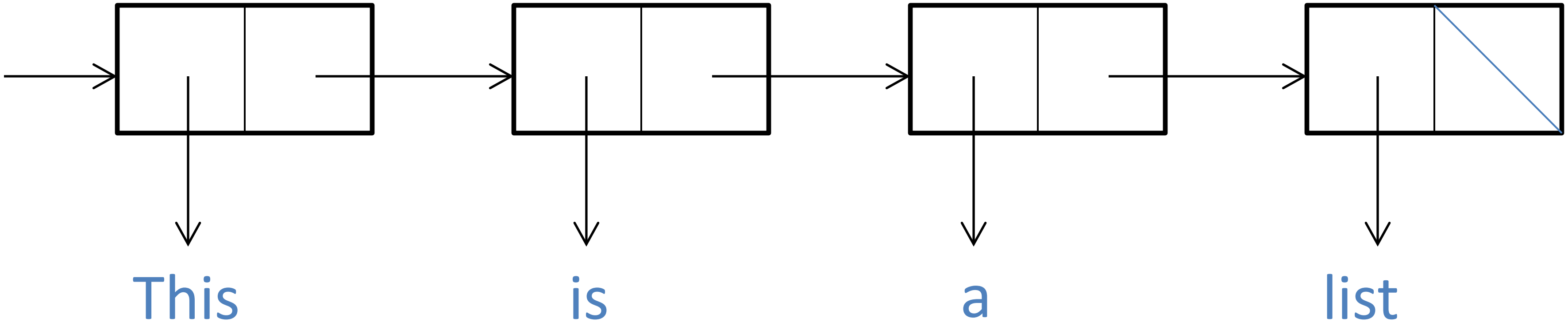
Amortized Analysis

- To perform $n=2^k$ adds to the end of an array takes O(n) time.
- What about $n \neq 2^{k'}$?
- Find the smallest $n'=2^k$ such that $n' \geq n$. Notice that $n>n'/2$.
- n adds take less time than n' adds, which take $O(n')=O(2n)=O(n)$ time.

Amortized Analysis

- So any n adds to the end of an array take O(n) time.
- On average, add operations take O(1) time!

Linked Lists



This

is

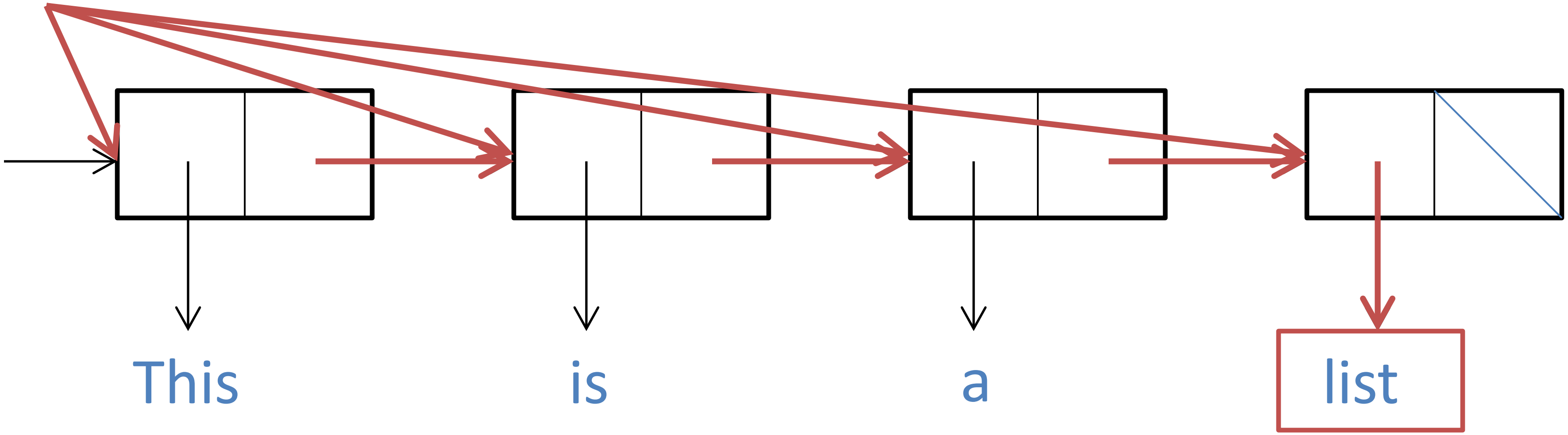
a

list

Linked Lists

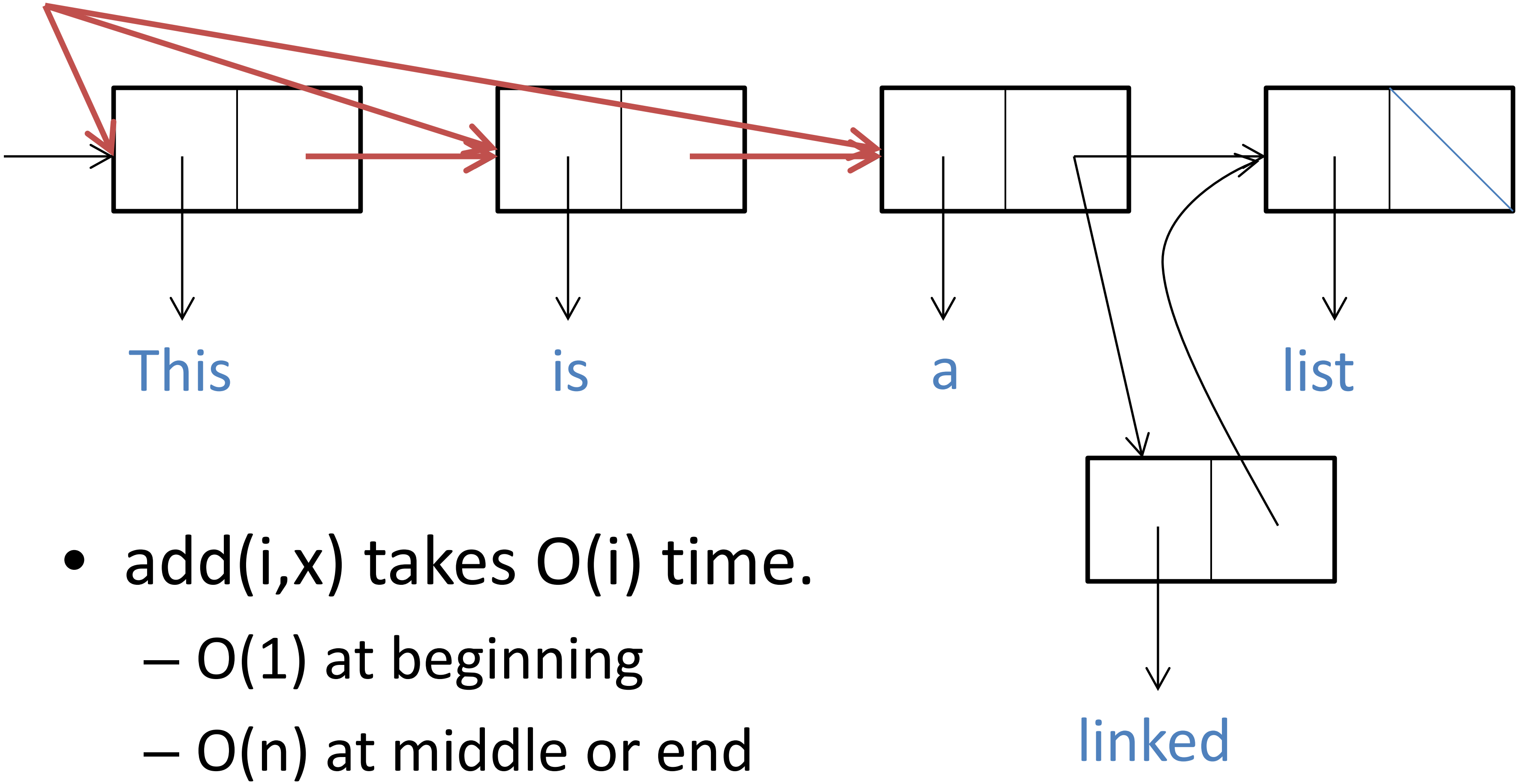
- Operations:
  - get(i) returns the value at index i
  - add(i,x) inserts x at index i
    - All values after index i get higher index
  - remove(i) removes the value at index i
    - All values after index i get lower index

get(4)



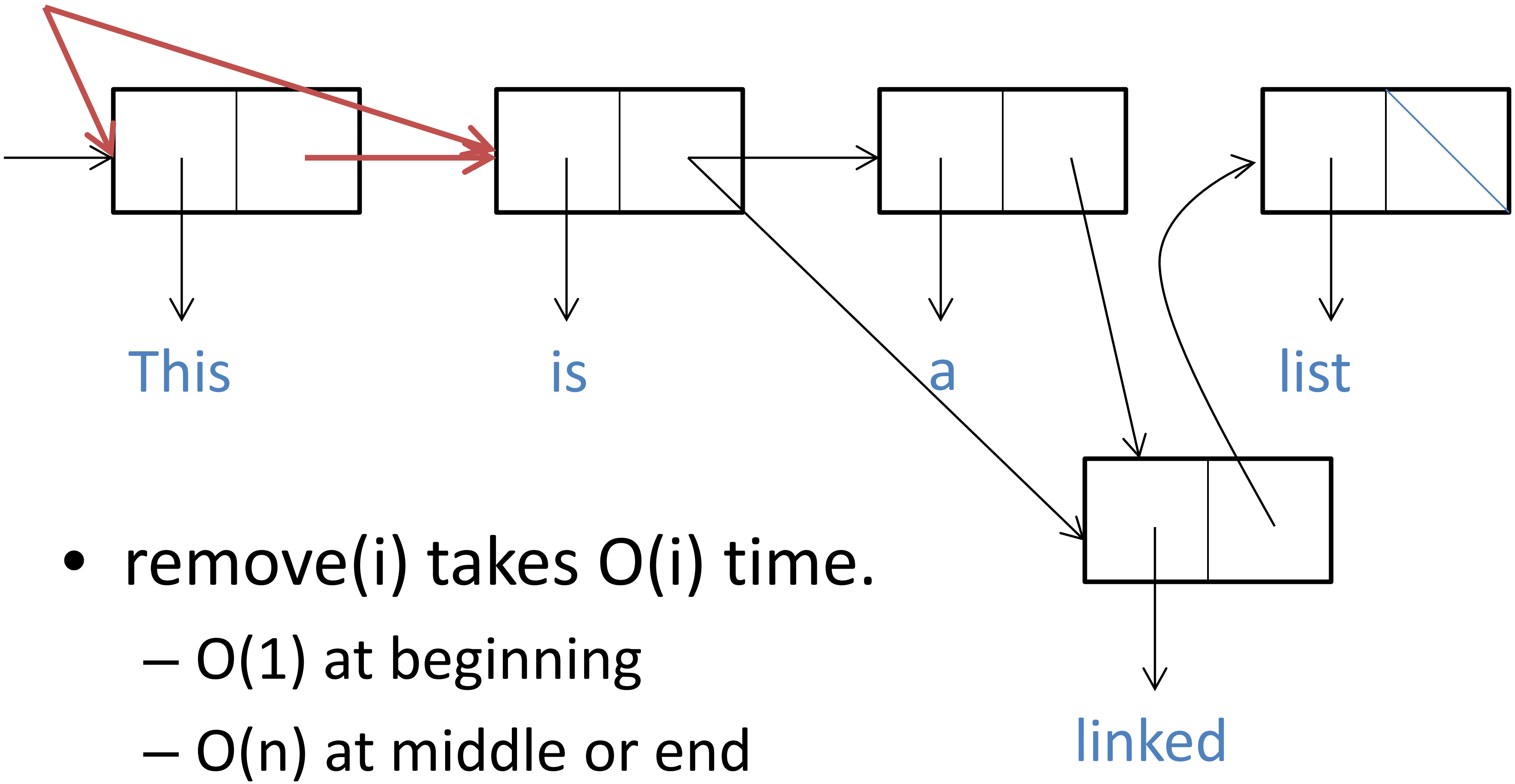
- get(i) takes O(i) time
  - O(1) at beginning
  - O(n) at middle or end

add(3,"linked")



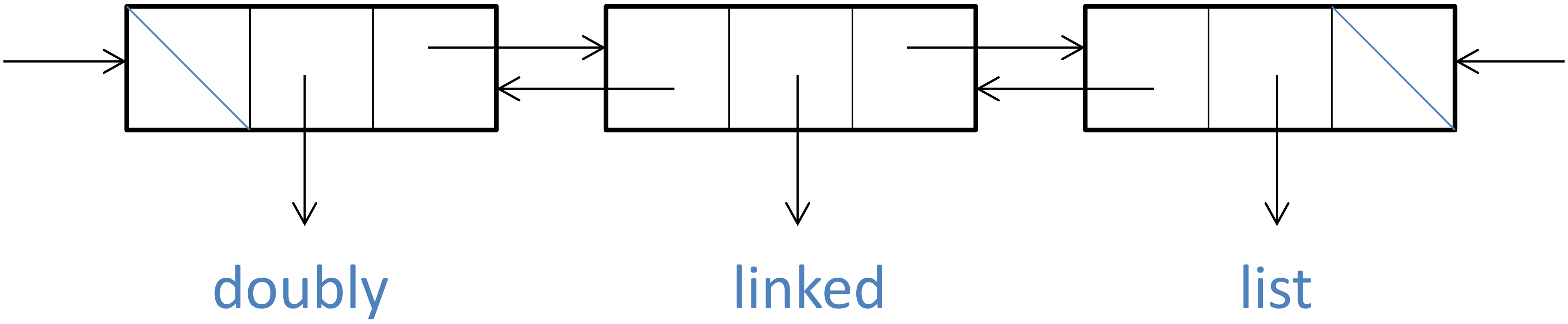
- add(i,x) takes O(i) time.
  - O(1) at beginning
  - O(n) at middle or end

remove(2)



- remove(i) takes O(i) time.
  - O(1) at beginning
  - O(n) at middle or end

Optimization: Doubly Linked List



- add(i,x), get(i), remove(i) take O(min(i,n-i)) time
  - O(1) at beginning or end
  - O(n) in the middle

Arrays vs. Lists

	add, remove			**get**		
	Beginning	Middle	End	Beginning	Middle	End
Array	O(n)	O(n)	O(1)*	O(1)	O(1)	O(1)
Linked List	O(1)	O(n)	O(n)	O(1)	O(n)	O(n)
Doubly Linked List	O(1)	O(n)	O(1)	O(1)	O(n)	O(1)

Stacks

- Last in, first out (LIFO) behavior
- Operations:
  - push(x): adds element x to top of stack
  - peek(): returns top element of stack w/o removal
  - pop(): removes and returns top of stack

Implementing Stacks

- With a linked list:
  - push(x) = add(0,x) O(1)
  - peek() = get(0) O(1)
  - pop() = { O(1)

```
pop() = {
    temp = get(0)
    remove(0)
    return temp
}
```

Implementing Stacks

- With a dynamic array:
  - push(x) = add(n,x) O(1)
  - peek() = get(n-1) O(1)
  - pop() = { O(1)

    ```
        temp = get(n-1)
        remove(n-1)
        return temp
    }
    ```

Queues

- First in, first out (FIFO) behavior
- Operations:
  - add(x): adds element to end of queue
  - peek(x): returns head of queue w/o removal
  - poll(x): removes and returns head of queue

Implementing Queues

- With a double linked list:
  - add(x) = add(0,x) O(1)
  - peek() = get(n-1,x) O(1)
  - poll() = { O(1)

    ```
        temp = get(n-1)
        remove(n-1)
        return temp
    }
    ```

Implementing Queues

- With a dynamic array?
  - add(x) = add(n-1,x) O(1)
  - peek() = get(0) O(1)
  - poll() = {

```
    temp = get(0)
    remove(0)
    return temp
}
```

Dictionaries

- In list structures, we look up value by its index
- What if we want to look up value by some other key?
- Example:
  - Want to store GPAs of all students, organized by student name.

Dictionaries

- Desired Operations:
  - add(key,value): associates value to key
  - lookup(key): returns the value associated to key
  - remove(key): removes key and corresponding value from dictionary

Implementing Dictionaries

- With a dynamic array:
  - add(key,value) = add(n, (key,value)) O(1)
    - Duplicates?
  - lookup(key) = { O(n)

```
    For pair (key’,value’) in list:
        If key’=key, return value’
}
```

  - remove(key) = { O(n)

```
    For pair (key’,value’) in list:
        If key’=key, remove this pair
}
```

Problem

- If we want constant time operations, we need an array
- Arrays indexed by integers, we want indexed by keys
- Even if keys are integers, we may want to allow keys much larger than the size of the array

Idea: Hash Tables

- Let K be the space of possible keys
- Let {0,…,n-1} be the possible indices of a length-n array
- Choose a function h: K → {0,…,n-1}
  - Called a hash function

Idea: Hash Tables

- Choose a function h: K → {0,…,n-1}
- add(key,value): put value into index h(key) O(1)
- lookup(key): get value at h(key) O(1)
- remove(key): delete value at h(key) O(1)

Problem

- What if h(key) = h(key’)?
  - We call this a collision
- Solution: Instead of storing value at each index, store a (linked or array) list of values

Chaining

- Each index of the array points to a linked list of (key,value) pairs.
- add(key,value) = {
  - Compute h(key), and let L be the list stored at the index h(key).
  - Search L for a pair (key’,value’) with key=key’
  - If pair found, replace with (key,value)
  - Otherwise add (key,value) to end.

  }

Chaining

- Similar operation for lookup and get
- Running time:
  - Must scan entire list, so all operations O(|L|)

Picking a good function

- All operations proportional to size of the chains
- To make efficient, need chains to be small (preferably constant size)

Idea 1: One-to-one function

- Ideally want $h(\text{key}_1) \neq h(\text{key}_2)$ for all $\text{key}_1 \neq \text{key}_2$
- Example: K = strings with m characters
  - Let space = 0, a = 1, b = 2, … , A = 27, B = 28, ...
  - Max value of character: 52
  - If $s=s_1 s_2 \ldots s_m$, then $h(s) = s_m + 53 s_{m-1} + 53^2 s_{m-2} + \ldots + 53^{m-1} s_1$
- Problem: need $n \geq 53^m$

Idea 2: Many-to-one

- Pick some function h that maps the same number of keys to each index
- Example: K = m-bit integers
  - Let h(x) = x mod n
- Problem:
  - What if I happen to store a bunch of values that map to the same index? (ex: 0, n, 2n, …)
  - Solution: randomness!

Idea 3: Random Functions

- Let h(key) be a random value for each key
- Problem:
  - Need every evaluation of h(key) to return the same value
  - Must remember h(key) for further evaluations
  - Looking up h(key) requires an efficient dictionary!

Idea 4: Universal Hashing

- To minimize collisions, truly random functions are overkill
- Pick from small set of functions that “appear” random

Universal Hashing

- Let H be some subset of the functions from K to {0,…,n-1} with the following property:
  - For all $key_1 \neq key_2$

$$\Pr_{h \leftarrow H}[h(key_1) = h(key_2)] \leq \frac{1}{n}$$

Example

- Say n is prime.
- Assume keys are integers smaller than n^k
- Pick k+1 random values in {0,…,n-1}: a_0, …, a_k
- Interpret key as k-digit number base n:
  - key = $n^{k-1}\, b_{k-1} + \ldots + k\, b_1 + b_0$
- h(key) = $b_0 a_0 + \ldots + b_{k-1} a_{k-1} + a_k$ mod n

Example

- Proof of universality:
  - Let key $\neq$ key'. They differ in some digit i (i.e. $b_i \neq b_i'$)
    - Assume w.l.o.g. that i=0
  - Then $h(key) - h(key') = (b_0 - b_0')a_0 + \ldots + (b_{k-1} - b_{k-1}')a_{k-1} \bmod n$
  - If h(key) = h(key'), then

$$a_0 = -\left((b_1 - b_1')a_1 + \ldots + (b_{k-1} - b_{k-1}')a_{k-1}\right)\left(b_0 - b_0'\right)^{-1} \bmod n$$

  - Happens for exactly one choice of a_0, prob = 1/n

Universal Hashing and Chaining

- Theorem: S be any subset of K, key an element of K not in S. Then if h is drawn from a universal family of hash functions,

$$E_{h \leftarrow H}[\text{number of } s \text{ where } h(key) = h(s)] \leq \frac{|S|}{n}$$

Universal Hashing and Chaining

$$E_{h\leftarrow H}[\text{number of } s \text{ where } h(key) = h(s)] \le \frac{|S|}{n}$$

- Proof: Let c_s be 0 if h(key) ≠ h(s), 1 otherwise

$$E_{h\leftarrow H}[c_s] = \Pr_{h\leftarrow H}[h(key) = h(s)] \le 1/n$$

$$E_{h\leftarrow H}[\text{number of } s \text{ where } h(key) = h(s)]$$

$$= E_{h\leftarrow H}\left[\sum_{s\in S} c_s\right] = \sum_{s\in S} E_{h\leftarrow H}[c_s] \le \frac{|S|}{n}$$

Universal Hashing and Chaining

- Let S be the set of values stored in hash table
- Key maps to chain of expected size 1 + |S|/n
- If we keep |S|/n constant (i.e. |S|/n ≤ 1), all operations constant time
- What happens if |S| gets to large?
  - Double size of array, choose new hash function, and move over all data to new array.
  - Expensive, but amortized constant time.

Running Times

- add: O(1) expected amortized time
- lookup: O(1) expected time
- remove: O(1) expected time