CS 161: Design and Analysis of Algorithms

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Outline

- Why study algorithms?
- What makes a good algorithm?
- What We'll See
  - Example: Multiplication
- Asymptotics review
- Course information

Why Study Algorithms?

- Required for CS majors
- Algorithms are everywhere in CS
  - OS (CS 140)
  - Compilers (CS 143)
  - Crypto (CS 155)
  - Etc.
- Algorithms important in other fields
  - Economics (Game Theory)
  - Biology
- Exciting!

What Makes a Good Algorithm

- Computes desired result
  - Always?
  - With high probability?
  - On real-world inputs?
- Uses resources efficiently
  - Time?
  - Space?
  - Disk I/O?
  - Programmer Effort?

CS 161 Concepts

- Data Structures
- Graph Algorithms
- Greedy Algorithms
- Divide & Conquer
- Dynamic Programming
- Linear Programming
- NP-Completeness

Algorithms We’ll See

- Integer/Matrix Multiplication
- Fast Fourier Transform (FFT)
- Shortest Paths
- Sequence Alignment
- Minimum Spanning Trees
- Maximum Flows

Integer Multiplication

- Grade-school algorithm

$$\begin{array}{r} 3764 \\ \times\ 689 \\ \hline 33876 \\ 301120 \\ +\ 2258400 \\ \hline 2593396 \end{array} \qquad \begin{array}{r} 3764 \\ \times\ 9 \\ \hline 33876 \end{array} \qquad \begin{array}{r} 3764 \\ \times\ 8 \\ \hline 30112 \end{array}$$

$$\begin{array}{r} 3764 \\ \times\ 6 \\ \hline 22584 \end{array}$$

Can we do better? Yes!

Matrix Multiplication

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} \times \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \cdots & b_{n,n} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n,1} & c_{n,2} & \cdots & c_{n,n} \end{pmatrix}$$

$$c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}$$

- Computing each term uses n multiplications and (n-1) additions
- Total: ~$2n^3$ operations

Can we do better? Yes!

Discrete Fourier Transform (DFT)

- Given a sequence $a=(a_1, a_2, \ldots, a_n)$, the DFT of a is defined as $A=(A_1, A_2, \ldots, A_n)$ where

$$A_k = \sum_{i=1}^{n} a_i {\omega_n}^{ik}$$

- Important in signal processing, solving differential equations, polynomial multiplication, and integer multiplication!

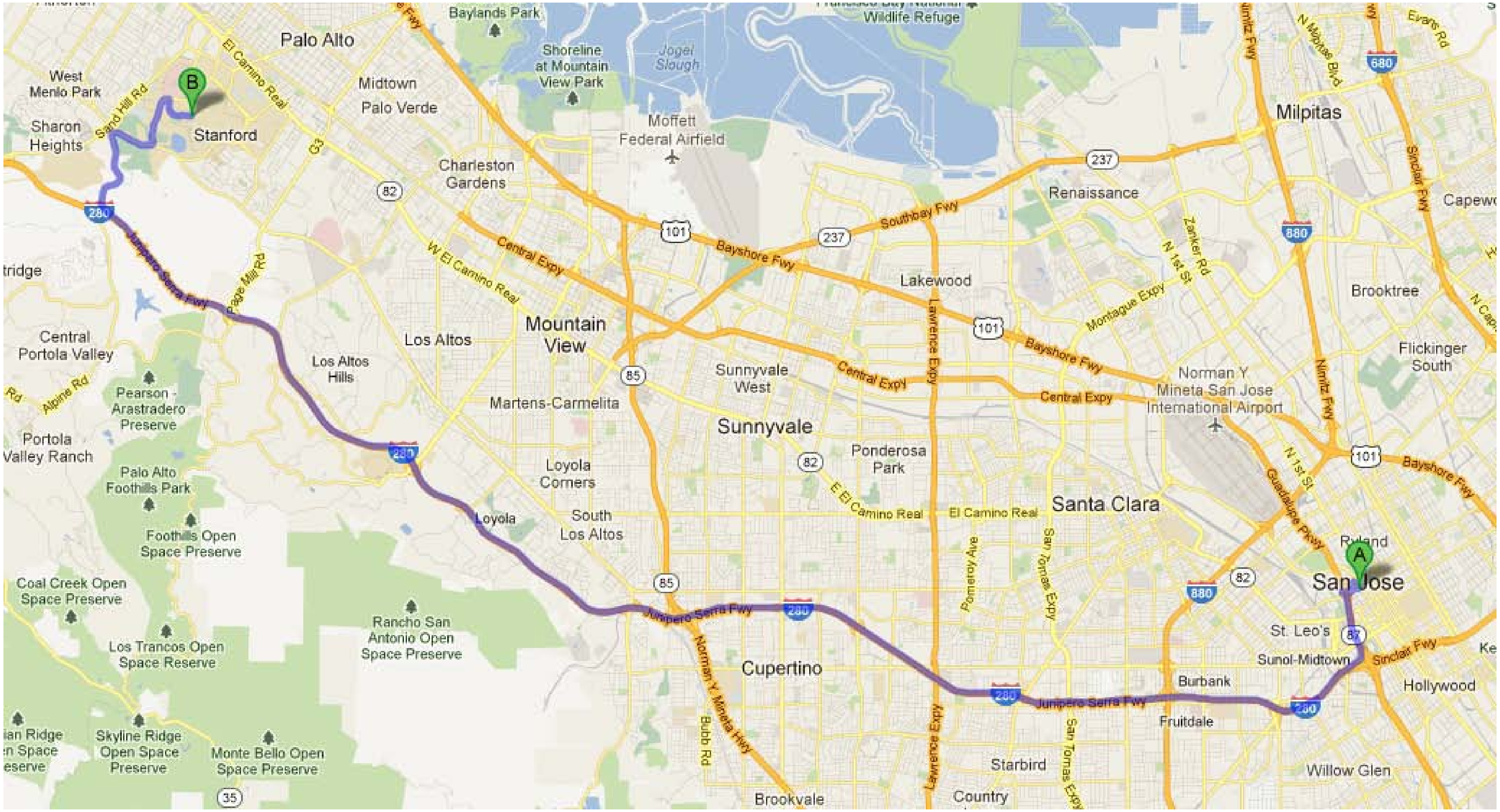
Discrete Fourier Transform (DFT)

$$A_k = \sum_{i=1}^{n} a_i \omega_n^{ik}$$

- Appears to require ~n^2 additions and multiplications.

Can we do better? Yes!

Shortest Paths



How do we determine the shortest path without exploring all paths?

Sequence Alignment

- How close is “snowy” to “sunny”?
  - Cost: 3 modifications:

```
S – N O W Y
S U N N – Y
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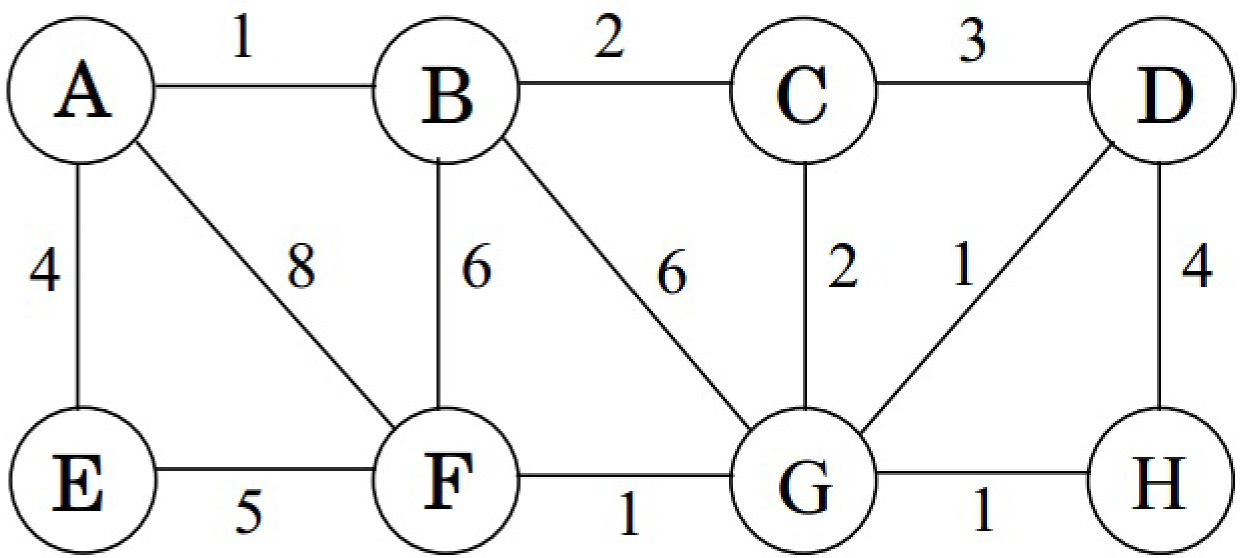
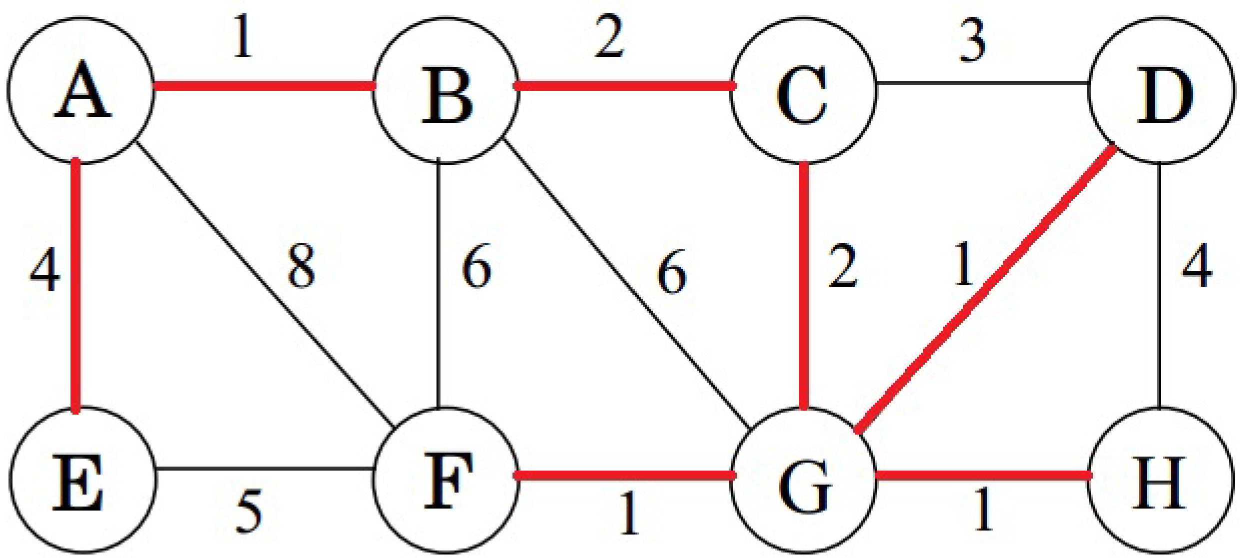
- Applications:
  - Spell-checkers: If I have a misspelled word, what did I mean?
  - Biology: Identify sections of DNA that are similar

Minimum Spanning Trees

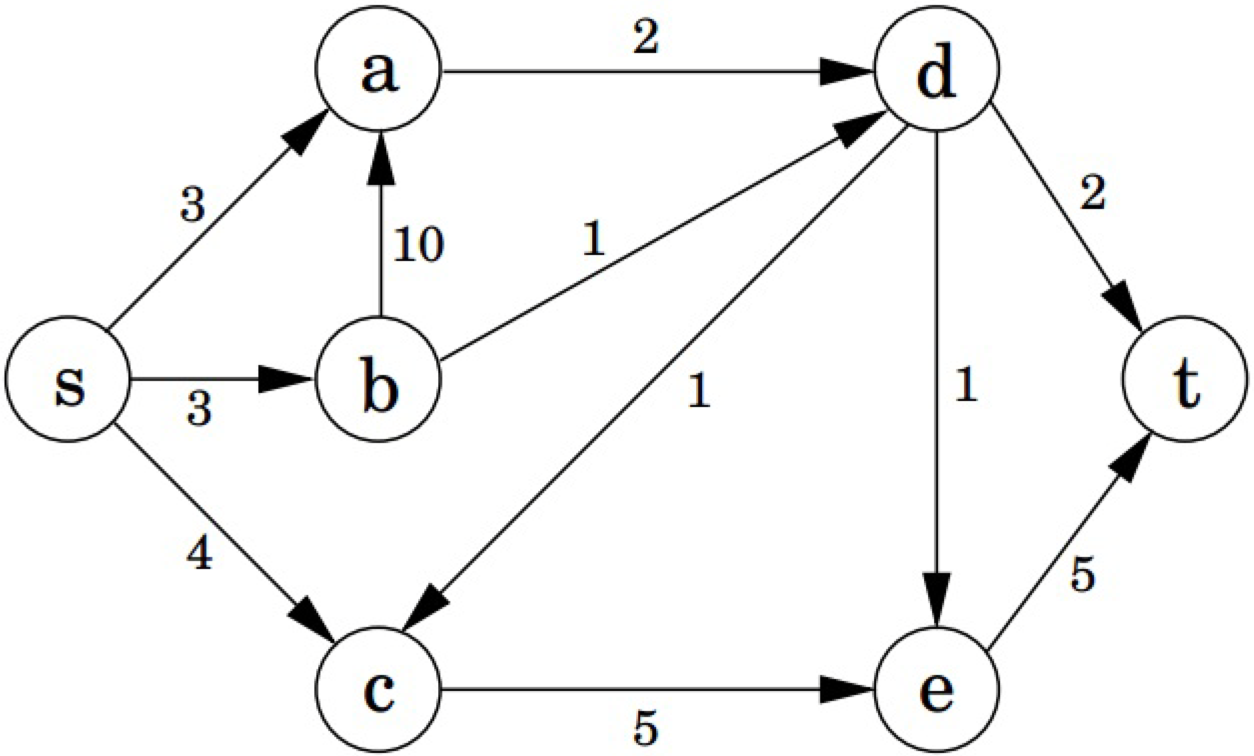


Minimum Spanning Trees

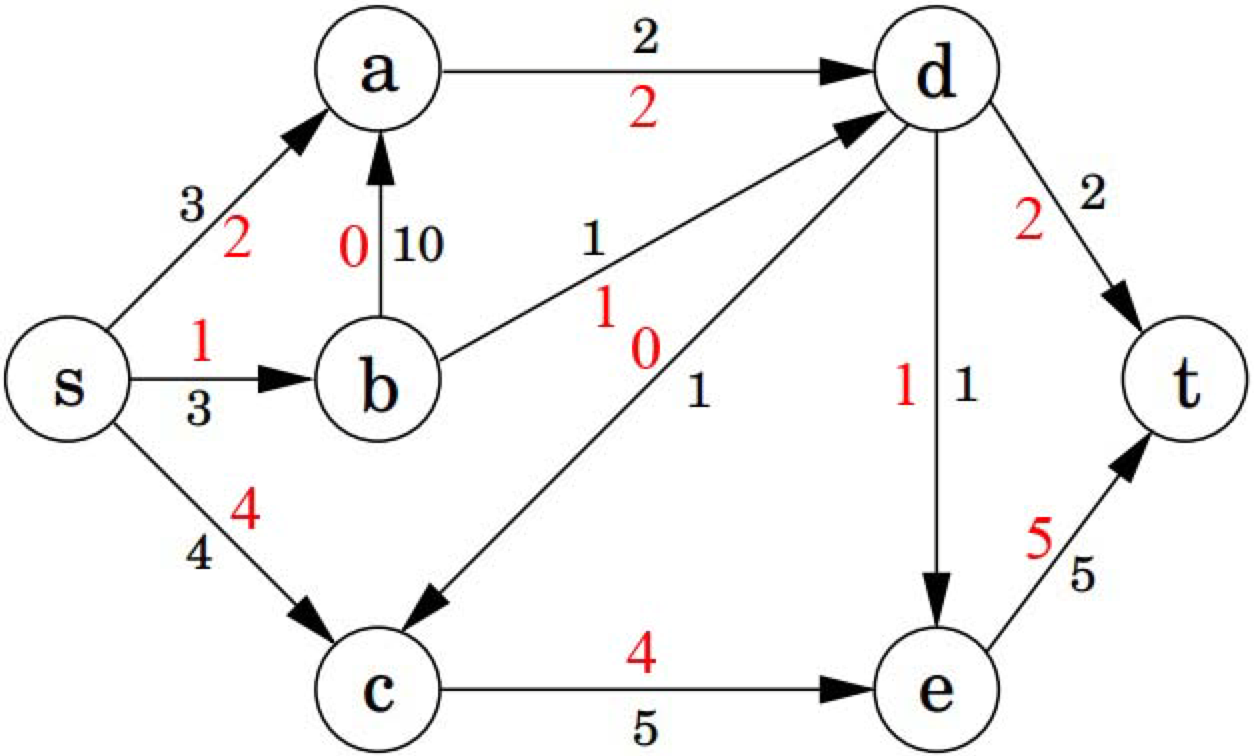


- Applications to network design, approximation algorithms, and more.

Maximum Flows



Maximum Flows



- Used in solutions for a host of different problems

Example: Integer Multiplication

- Algorithm 1: Grade-school algorithm

$$\begin{array}{r} 3764 \\ \times\ 689 \\ \hline 33876 \\ 301120 \\ +\ 2258400 \\ \hline 2593396 \end{array}$$

$$\begin{array}{r} 3764 \\ \times\ 9 \\ \hline 33876 \end{array} \qquad \begin{array}{r} 3764 \\ \times\ 8 \\ \hline 30112 \end{array}$$

$$\begin{array}{r} 3764 \\ \times\ 6 \\ \hline 22584 \end{array}$$

Example: Integer Multiplication

- Algorithm 1: Grade-school algorithm
  - For simplicity, assume running time dominated by single-digit multiplications
  - Number of single-digit multiplications: n^2
  - Can we do better?

Example: Integer Multiplication

- Algorithm 2: Recursive algorithm
  - Write n-digit number x as $10^{n/2} x_1 + x_2$
    - x_1, x_2 are n/2-digit numbers
  - Write y as $10^{n/2} y_1 + y_2$
  - $xy = 10^n x_1y_1 + 10^{n/2}(x_1y_2 + x_2y_1) + x_2y_2$
  - Can multiply n-digit numbers by performing 4 multiplications of n/2-digit numbers.

Example: Integer Multiplication

- Algorithm 2: Recursive algorithm
  - $xy = 10^n x_1y_1 + 10^{n/2}(x_1y_2 + x_2y_1) + x_2y_2$
  - Claim: the number of single-digit multiplications for algorithm 2 is n^2.
    - True for n=1
    - Assume true for n/2. Then #(multiplications for n-bits) = 4 #(multiplications for n/2 bits) = $4\ (n/2)^2 = n^2$.

Example: Integer Multiplication

- Which algorithm is better?
  - Both require n^2 single-digit multiplications, so running time almost the same

Example: Integer Multiplication

- Algorithm 3: Another Recursive Algorithm
  - We need the quantities x_1y_1, $(x_1y_2 + x_2y_1)$, and x_2y_2
  - Can we compute $x_1y_2 + x_2y_1$ using one multiplication?
  - Gauss: $x_1y_2 + x_2y_1 = (x_1 + x_2)(y_1 + y_2) - x_1y_1 - x_2y_2$
    - Already have x_1y_1 and x_2y_2!

Example: Integer Multiplication

- Algorithm 3: Another Recursive Algorithm
  - Compute $P = x_1 y_1$, $Q = x_2 y_2$.
  - Let $x_3 = x_1 + x_2$, $y_3 = y_1 + y_2$.
  - Compute $R = x_3 y_3$
  - Let $S = R - P - Q$.
  - $xy = 10^n P + 10^{n/2} S + Q$

Example: Integer Multiplication

- Algorithm 3: Another Recursive Algorithm
  - Replaced 4 multiplications with 3, seems like it should be faster.
  - Not so simple:
    - One of the multiplications is slightly larger.
    - Added some extra overhead (additions and subtractions)
  - Possible to show $< 4n^{1.6}$ single-digit multiplications

Example: Integer Multiplication

- Which algorithm is faster?
  - $4n^{1.6} > n^2$ for $n < 32$.
  - $4n^{1.6} < n^2$ for $n > 32$.
- Algorithm 3 slower for $n < 32$, faster for $n>32$.
- How do we compare?

How to Compare Algorithms

- Say we want to compute a function f(n)
  - Algorithm 1 runs in 100,000n seconds
  - Algorithm 2 runs in n^2 seconds
- Which algorithm is better?
  - Algorithm 2 runs faster if $n<100,000$
  - Algorithm 1 runs faster if $n>100,000$
  - The ratio of run times goes to infinity

How to Compare Algorithms

- Say we want to compute a function g(n)
  - Algorithm 1 runs in 20n seconds
  - Algorithm 2 runs in 10n+30 seconds
- Which algorithm is better?
  - Algorithm 2 is faster than Algorithm 1 when n>3
  - However, it is only twice as fast

Asymptotics/Big Oh

- We say Algorithm 1 is “at least as fast as” Algorithm 2 if, for large enough inputs, Algorithm 1 is at most a constant factor slower than Algorithm 2.
- For this class, we will compare algorithms using “at least as fast as”.

Big Oh

$$O(f(n)) =$$

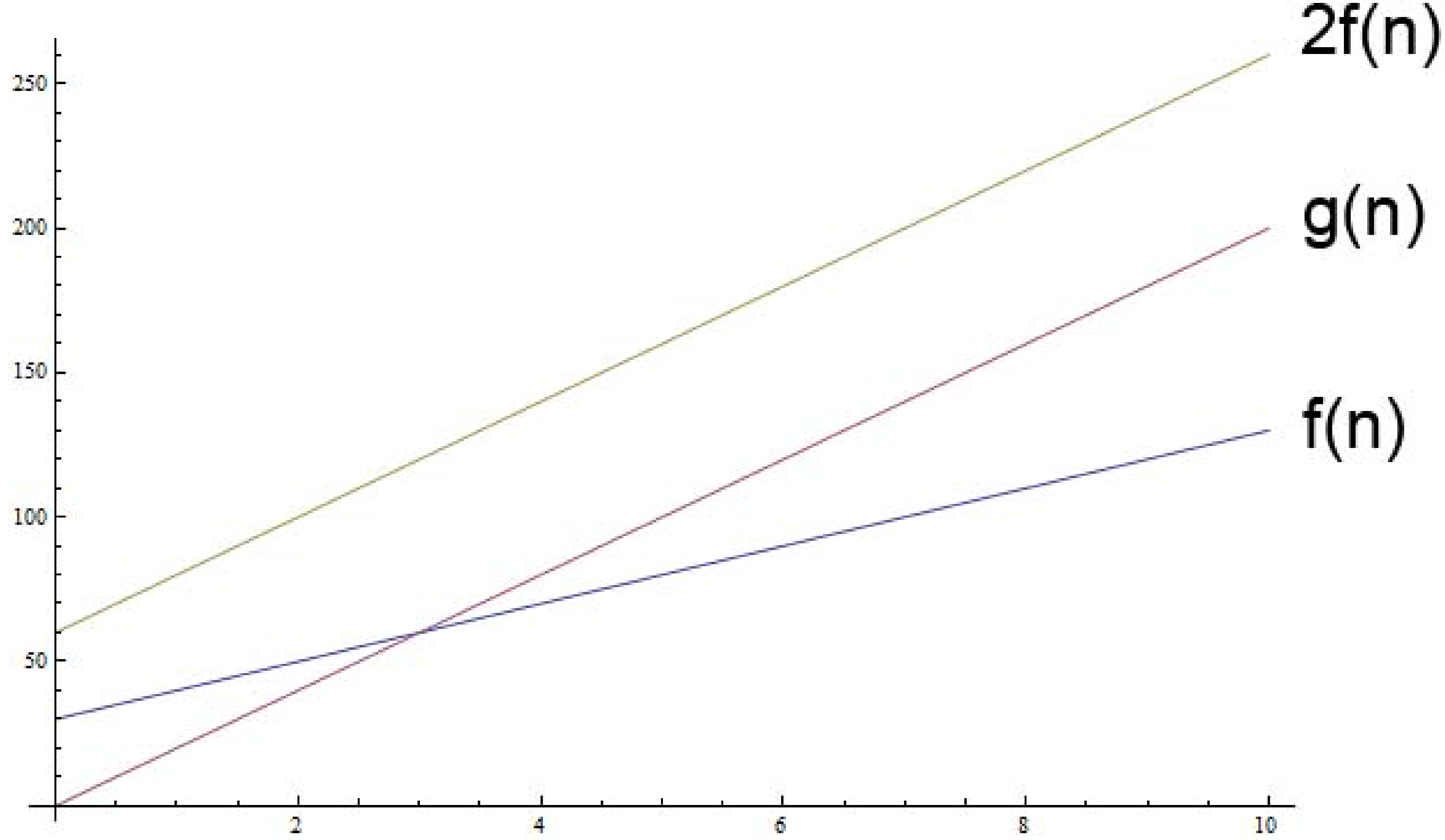
$$\{g(n) : \exists c, n_0 \text{ such that } g(n) \leq c\, f(n) \forall n \geq n_0\}$$

- So g(n) ∈ O(f(n)) if
  - There are c and n_0 such that
  - g(n) ≤ c f(n) for all n ≥ n_0.
- Sometimes write g(n)=O(f(n)).

Big Oh Example 1

- $f(n) = 10n+30$, $g(n) = 20n$
- $f(n) \in O(g(n))$
  - Proof: Let $c = 1$, $n_0 = 3$.
  - If $n \geq n_0$, then $f(n)=10n+30 \leq 20n = c\ g(n)$.
- $g(n) \in O(f(n))$
  - Proof: Let $c = 2$, $n_0 = 1$.
  - If $n \geq n_0$, then $g(n)=20n \leq 2 \times (10n+30) = c\ f(n)$.

Big Oh Example 1



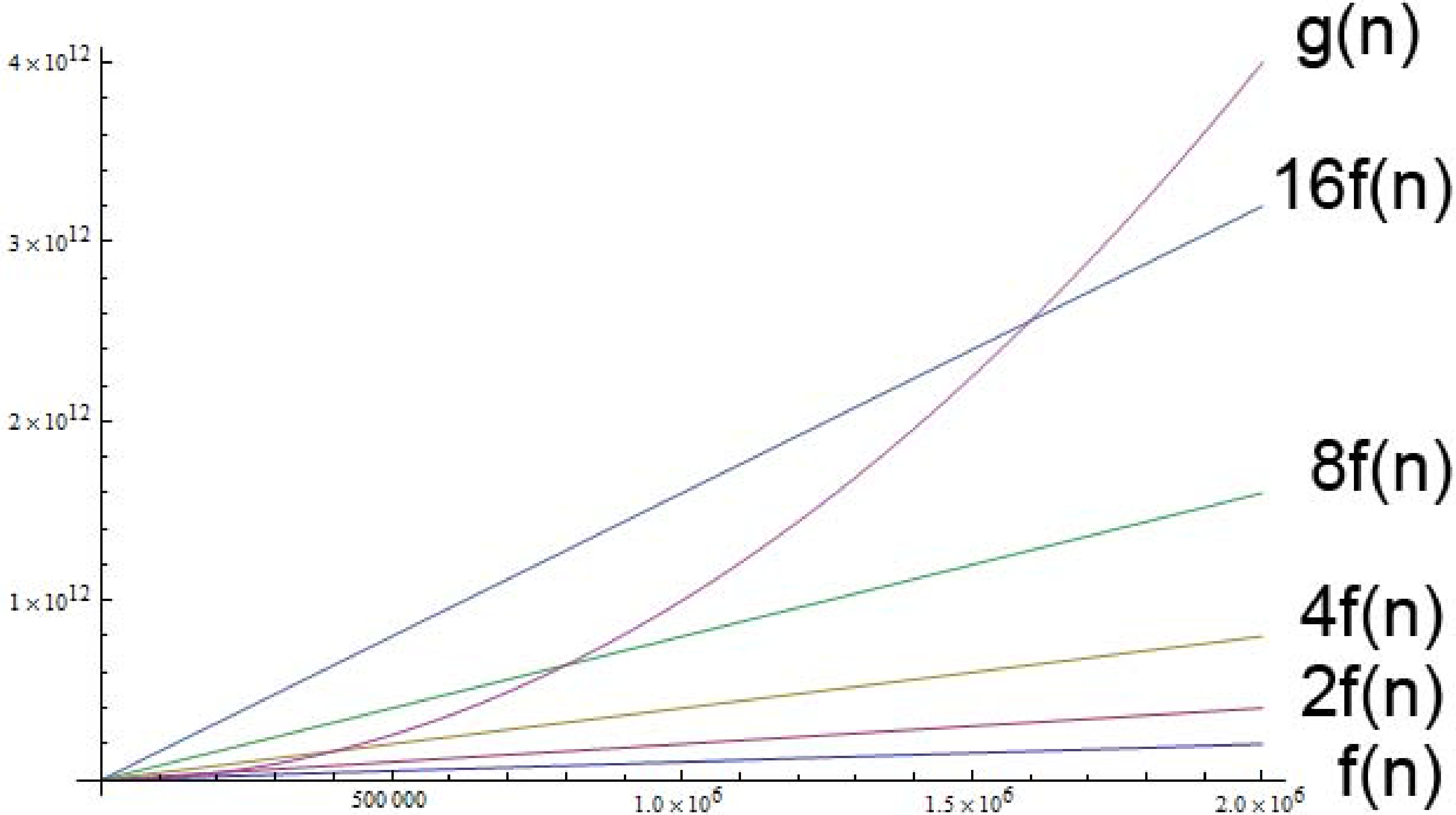
Big Oh Example 2

- $f(n) = 100{,}000n$, $g(n) = n^2$.
- $f(n) \in O(g(n))$
  - Proof: Let $c = 1$, $n_0 = 100{,}000$.
  - If $n \geq n_0$, then $f(n)=100{,}000n \leq n^2 = c\, g(n)$.
- $g(n) \in O(f(n))$?

Big Oh Example 2

- $f(n) = 100{,}000n$, $g(n) = n^2$.
- If $g(n) \in O(f(n))$, then
  - There are c, n_0 such that
  - $g(n) \le c\, f(n)$ for all $n \ge n_0$.
- But, for any c, suppose $n > 100{,}000c$.
  - Then $g(n) = n^2 > 100{,}000c\, n = c\, f(n)$
- So $g(n) \notin O(f(n))$.

Big Oh Example 2



Big Oh Facts

- Big Oh allows us to ignore constant factors:
  - For any constant c, $c\, f(n) \in O(f(n))$
- Big Oh allows us to ignore lower order terms:
  - If $f(n) \in O(g(n))$, then $g(n)+f(n) \in O(g(n))$

Big Oh Facts

- If $a \leq b$, then $n^a \in O(n^b)$
  - Proof: $n^a \leq n^b$ for all $n \geq 1$.
- If $1<a \leq b$, then $a^n \in O(b^n)$
  - Proof: $a^n \leq b^n$ for all $n > 0$.
- For any a,b, $\log_a(n) \in O(\log_b(n))$
  - Proof: $\log_a(n)=\log_b(n) \log_a(b)$ for all $n > 0$.

We can usually ignore the base for logarithms.
We will use $\log(n)$ to denote $\log_2(n)$

Big Oh Facts

- For any $a>0$, $b>1$, $n^a \in O(b^n)$
  - We'll see a proof in a bit
- For any $a>0$, $\log n \in O(n^a)$
  - We'll see a proof in a bit

Big Oh Facts

- If $f(n) \in O(g(n))$, and $g(n) \in O(h(n))$, then

  $$f(n) \in O(h(n))$$

  – Proof: There are c, n_0 such that $f(n) \leq c\, g(n)$ for $n \geq n_0$.

  – There are c', n_0' such that $g(n) \leq c'\, f(n)$ for $n \geq n_0'$.

  – Let $c''=c\, c'$, $n_0''=\mathrm{Max}(n_0, n_0')$.

  – If $n \geq n_0''$, then $f(n) \leq c\, g(n) \leq c\, c'\, h(n) = c''\, h(n)$.

- $f(n) \in O(f(n))$: let $c=1$, $n_0=1$.

Big Oh inclusion is a sort of "$\leq$" on functions

Big Omega

$$\Omega(f(n)) =$$

$$\{g(n) : \exists c, n_0 \text{ such that } g(n) \geq c\, f(n) \forall n \geq n_0\}$$

- So $g(n) \in \Omega(f(n))$ if
  - There are c and n_0 such that
  - $g(n) \geq c\, f(n)$ for all $n \geq n_0$.
- Sometimes write $g(n)=\Omega\,(f(n))$.

Big Omega

- $g(n) \in \Omega(f(n))$ is equivalent to $f(n) \in O(g(n))$
  - Proof: $f(n) \in O(g(n))$ means there is c, n_0 such that
    $f(n) \leq c\, g(n)$ for $n \geq n_0$.
  - Then $g(n) \geq (1/c)\, f(n)$ for $n \geq n_0$.
  - Similar proof for other direction.

Big Omega inclusion is a sort of "≥" on functions

Big Theta

- $g(n) \in \Theta(f(n))$ if:
  - $g(n) \in O(f(n))$ and
  - $g(n) \in \Omega(f(n))$ (or equivalently $f(n) \in O(g(n))$)
- Note: the constants c, n_0 may be different for showing $g(n) \in O(f(n))$ and $g(n) \in \Omega(f(n))$
- $g(n) \in \Theta(f(n))$ is equivalent to $f(n) \in \Theta(g(n))$

Big Theta inclusion is a sort of "=" on functions

Little o, omega

- $f(n) \in o(g(n))$ if, for all c, there is an n_0:

  $f(n) < c\, g(n)$ for all $n \geq n_0$.
- $f(n) \in \omega(g(n))$ if, for all c, there is an n_0:

  $f(n) > c\, g(n)$ for all $n \geq n_0$.
- $f(n) \in o(g(n))$ is equivalent to $g(n) \in \omega(f(n))$

Little o and omega are a sort of “<” and “>” for functions

Big Oh Is Not Total!

- It is not true that $f(n) \in O(g(n))$ or $g(n) \in O(f(n))$
  - $f(n) = n^{1+\mathrm{Cos}(\pi n)}$, $g(n) = n$
- Similarly, $f(n) \notin O(g(n))$ does not imply that $f(n) \in \omega(g(n))$

Limit Test

- Let $c = \lim_{n \to \infty} f(n)/g(n)$
  - If $c = 0$, then $f(n) \in o(g(n))$
  - If $0 < c < \infty$, then $f(n) \in \Theta(g(n))$.
  - If $c = \infty$, then $f(n) \in \omega(g(n))$

Limit Test

- For any $a>0$, $\log n \in o(n^a)$
  - Proof: Let $c = \lim_{n\to\infty} \log n/n^a$
  - L'Hôpital's rule:

    $c = \lim_{n\to\infty}(1/n)/(a\, n^{a-1}) = \lim_{n\to\infty} 1/(a\, n^a) = 0.$

Limit Test

- For any $a>0$, $b>1$, $n^a \in o(b^n)$
  - Proof: Let $c = \lim_{n\to\infty} n^a/b^n$
  - L'Hôpital's rule:

    $c = \lim_{n\to\infty} (a\, n^{a-1})/(b^n \ln b) = (a/\ln b) \lim_{n\to\infty} n^{a-1}/b^n$
  - Repeat until $a \leq 0$
  - We have that $c = \text{const} \times \lim_{n\to\infty} n^{a'}/b^n$ for $a' \leq 0$
    - Numerator goes to 0, denominator goes to infinity
    - Thus $c = 0$.

Other Notation

- $f(n) \in g(n)+o(h(n))$ means $f(n)-g(n) \in o(h(n))$
  – Common example: $f(n) \in g(n)+o(1)$ means $f(n)$ converges to $g(n)$.
- $f(n) \in g(n)^{O(h(n))}$ means there is some function $h'(n) \in O(h(n))$ such that $f(n)=g(n)^{h'(n)}$.
  – Common example: $f(n) \in n^{O(1)}$ means $f(n) \in O(n^a)$ for some a.

Course Information

- Lectures: WMF 2:15-3:30
- Text: *Algorithm Design* by Kleinberg and Tardos
- Webpage: http://cs161.stanford.edu
- Piazza: http://piazza.com/class#summer2012/cs161
- Staff Email: cs161-summer2012-staff@lists.stanford.edu

Grading

- Homeworks: 50%
  - Five weekly homeworks
- Midterm: 20%
  - Wednesday, July 25^{th} in class
- Final: 30%
  - Friday, August 17^{th}, Location: TBA

Homework

- Assigned each Wednesday, due following Friday
  - Exceptions: no homework due week of midterm or final
  - Due at **beginning** of class (2:15PM)
- May work in groups of up to three
  - Write up solutions individually
- One 72-hour extension
- Can submit digitally to
  cs161-summer2012-submissions@lists.stanford.edu
- See website for policies