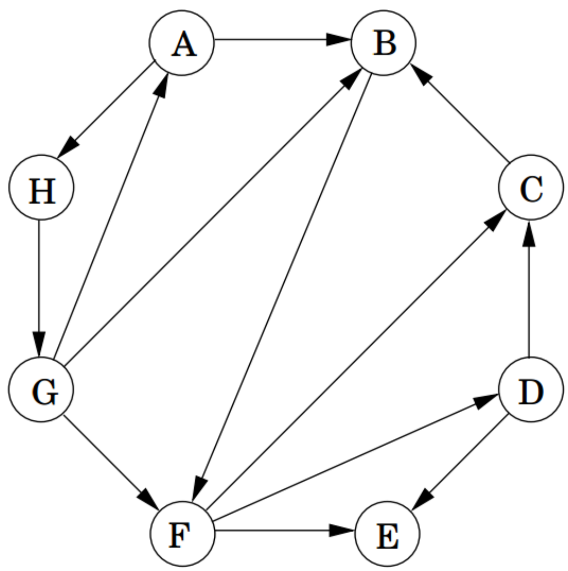
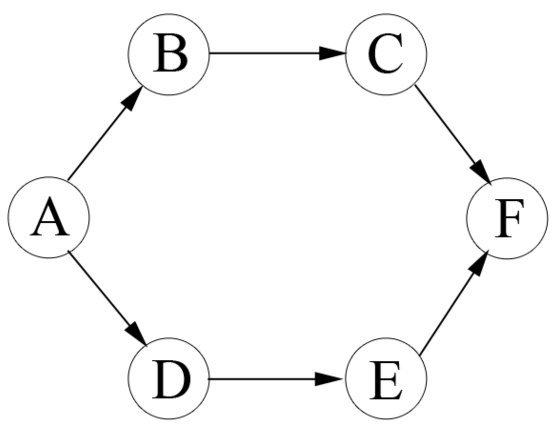
Problem 1 (10 Points). Perform a depth-first search on the following graph. Whenever there's a choice of vertices, pick the one that is alphabetically first. Classify each edge as a tree edge, forward edge, back edge, or cross edge, and give the `pre` and `post` number of each vertex (for this problem, you may scan handwritten drawings).



Problem 2 (5 Points). Consider the following DAG:



How many topological orderings does it have? List them.

Problem 3 (20 Points). A *bipartite* graph is an undirected graph $G = (V, E)$ whose nodes can be partitioned into two sets V_1 and V_2 (i.e. $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$) such that all edges have one endpoint in V_1 and the other endpoint in V_2.

(a) Give a linear-time algorithm to determine whether an undirected graph is bipartite. If the graph is bipartite, your algorithm should output V_1 and V_2.

(b) Prove that a graph is bipartite if and only if it has no cycles of odd length.

Problem 4 (10 Points). Suppose a CS curriculum consists of n courses, all of them mandatory. The prerequisite graph G has a node for each course, and an edge from course v to course w if and only if v is a prerequisite for w. Find an algorithm that works directly with this graph representation, and computes the minimum number of quarters necessary to complete the curriculum (assuming that a student can take any number of courses in one quarter). The running time of your algorithm should be linear. Prove its correctness and running time.

Problem 5 (10 Points). We have three containers whose sizes are 10 pints, 7 pints, and 4 pints, respectively. The 7-pint and 4-pint containers start out full of water, but the 10-pint container is initially empty. We are allowed one type of operation: pouring the contents of one container into another, stopping only when the source container is empty or the destination container is full. We want to know if there is a sequence of pourings that leaves exactly 2 pints in the 7- or 4-pint container.

(a) Model this as a graph problem: give a precise definition of the graph involved and state the specific question about this graph that needs to be answered.

(b) What algorithm should be applied to solve the problem?

Problem 6 (10 Points). The police department in the city of Computopia has made all streets one-way. The mayor contends that there is still a way to drive legally from any intersection in the city to any other intersection, but the opposition is not convinced. A computer program is needed to determine whether the mayor is right. However, the city elections are coming up soon, and there is just enough time to run a linear-time algorithm.

(a) Formulate this problem graph-theoretically, and explain why it can indeed be solved in linear time.

(b) Suppose it now turns out that the mayor's original claim is false. She next claims something weaker: if you start driving from town hall, navigating one-way streets, then no matter where you reach, there is always a way to drive legally back to the town hall. Formulate this weaker property as a graph-theoretic problem, and carefully show how it too can be checked in linear time.

Problem 7 (35 Points). When on your favorite social network, do you ever notice that your friends tend to have more friends than you do? Don't despair! There is a good reason that most people have fewer friends than there friends do, and it is known as the Friendship Paradox.

Let's model a social network as an undirected graph, with nodes being people, and edges being friendships. Then the degree $d(v)$ of a node v is the number of friends that person has. Let $f(v)$ be the average degree of the neighbors of v (i.e. the average number of friends v's friends have). That is

$$f(v) = \frac{1}{d(v)} \sum_{(v,u)\in E} d(u)$$

If a node v has no neighbors, we let $f(v) = 0$. Let D be the average of $d(v)$ over all v, and let F be the average of $f(v)$ over all v. We want to compare F to D.

(a) Show that $D = 2|E|/|V|$.

(b) Give an example of a family of graphs for $|V| = 2, 3, ...$ where $F = \Omega(D|V|)$.

(c) Give an example of a family of graphs for $|V| = 2, 3, ...$ where $F = D$.

(d) Show that

$$\frac{F}{D} = \frac{1}{2|E|} \sum_{(v,u)\in E} \left(\frac{d(u)}{d(v)} + \frac{d(v)}{d(u)}\right)$$

For positive x, $x + 1/x \geq 2$, with equality if and only if $x = 1$. (Proof: you can verify that $x + 1/x = 2 + \frac{(x-1)^2}{x} \geq 2$, with equality if and only if $(x-1)^2 = 0$, i.e. $x = 1$). Further, it is easy to see that $x + 1/x \leq \max(x, 1/x) + 1$.

(e) Use these facts to conclude that all graphs have $D \leq F \leq D|V|/2$.

(f) Precisely categorize which graphs have $F = D$ and which have $F > D$.

Total points: 100